Unboundness of the first eigenvalue of the Laplacian in symplectic category

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Abstract

Given a closed symplectic manifold (M^{2n}, ω) of dimension $2n \geqslant 4$, we consider all Riemannian metrics on M, which are compatible with the symplectic structure ω. For each such metric g, we look at the first eigenvalue λ_1 of the Laplacian associated with it. We show that λ_1 can be made arbitrarily large, when we vary g. This generalizes previous results of Polterovich, and of Mangoubi.

1 Introduction and main results

The current paper addresses the discussion on rigidity versus flexibility of the first eigenvalue of the Laplacian. The first result in this direction was proved by Hersch [He]:

Theorem 1.1. *Let (S^2, g) be the 2-sphere equipped with a Riemannian metric g. Then,*

$$\lambda_1(S^2, g) Area(S^2, g) \leqslant 8\pi,$$

where $\lambda_1(S^2, g)$ is the first positive eigenvalue of the Laplacian on (S^2, g).

In Theorem 1.1 the equality is known to occur if and only if (S^2, g) is the standard round sphere.

Hersh's theorem was extended to the case of a general closed surface, by Yang and Yau [Y-Y]:

Theorem 1.2. *Let (Σ, g) be a closed Riemannian surface. Then*

$$\lambda_1(\Sigma, g) Area(\Sigma, g) \leqslant 8\pi(genus(\Sigma) + 1).$$

[1] The author also uses the spelling “Buhovski” for his family name.

In Theorem 1.2, however, the upper bound is not optimal.

These results reflect a rigidity phenomenon in dimension 2, stating that in this case λ_1 is bounded, when we run over all Riemannian metrics that have a given volume. In contrast to the dimension 2, in higher dimensions, for the case of a "fixed volume form category", we have the following flexibility result of Colbois and Dodziuk [C-D]:

Theorem 1.3. *Let M be a closed manifold of dimension > 2, equipped with a volume form Ω. Consider the class of all Riemannian metrics on M having Ω as their volume form. Then this class admits metrics with arbitrarily large λ_1.*

However, if one restricts to a fixed conformal class of metrics, then we get a rigidity for λ_1, as the following result of El Soufi and Ilias [E-I], and of Friedlander and Nadirashvili [F-N] shows:

Theorem 1.4. *Let (M, g) be a closed Riemannian manifold of dimension d. Then*

$$\lambda_1(fg)Vol(M, fg)^{\frac{2}{n}} \leqslant C(g),$$

where f is any positive function on M and $C(g)$ is a constant independent of f.

The latter Theorem 1.4 can be seen as a generalization of Theorem 1.1, due to the Uniformization Theorem for the Riemann sphere.

As it turns out, Theorems 1.3 and 1.4 do not give us a full variety of ways in which one can generalize the 2-dimensional setting of Theorems 1.1 and 1.2. In [P], Polterovich proposes to look at a symplectic side of this story. For a given closed symplectic manifold (M, ω), he considers the Kahler, and the quasi-Kahler categories. In the Kahler case, Polterovich looks at the collection of all Riemannian metrics g on M, such that $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$, where J is a complex structure (i.e. an integrable almost complex structure) on M. In the quasi-Kahler case, Polterovich considers the collection of all Riemannian metrics g on M, such that $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$, where J is an almost (i.e. not necessarily integrable) complex structure on M. As it turns out, these two settings exhibit an opposite type of behaviour in terms of λ_1. Namely, in the Kahler case we meet with a rigidity phenomenon, whereas in the quasi-Kahler case we have examples of a flexibility, as the following two theorems of Polterovich show [P]:

Theorem 1.5. *Let (M, ω) be a closed symplectic manifold, such that ω is a rational form. Let g be a Kahler metric whose Kahler form is ω. Then*

$$\lambda_1 \leqslant C(\omega),$$

where $C(\omega)$ is independent of g.

Theorem 1.6. *Let $(\mathbb{T}^4, \sigma)$ be the standard symplectic 4-torus. Let (M, ω) be a closed symplectic manifold. Then, on $(\mathbb{T}^4 \times M, \sigma \oplus \omega)$ there exists a quasi-Kahler structure with arbitrarily large λ_1.*

In the case of the Kahler category, it is still an open question whether Theorem 1.5 is true for any closed symplectic manifold (M, ω). In the quasi-Kahler case, Theorem 1.6 was generalized by Mangoubi [M]:

Theorem 1.7. *Let $(\mathbb{T}^2, \sigma)$ be the standard symplectic 2-torus. Let (M, ω) be a closed symplectic manifold. Then, on $(\mathbb{T}^2 \times M, \sigma \oplus \omega)$ there exists a quasi-Kahler structure with arbitrarily large λ_1.*

In [M], Mangoubi raises the following conjecture:

Conjecture 1.8. *Let (M, ω) be a closed symplectic manifold of dimension $\geqslant 4$. Then, there exists a quasi-Kahler structure on it with arbitrarily large λ_1.*

The approach of Polterovich in the proof of Theorem 1.6 is based on a construction of an isotropic singular distribution on $(\mathbb{T}^4 \times M, \sigma \oplus \omega)$, which satisfies Hörmander condition. After providing the construction, Polterovich fixes some Riemannian metric on $\mathbb{T}^4 \times M$ which is compatible with $\sigma \oplus \omega$, and applies to it a "stretching the neck"-type procedure associated with the constructed distribution, and thus provides us with a new Riemannian metric on $\mathbb{T}^4 \times M$. Then finally Polterovich applies Hörmander theory [Ho] to show that by such a procedure one might get a desired Riemannian metric on $\mathbb{T}^4 \times M$, and this finishes the proof of Theorem 1.6.

Mangoubi, in order to prove Theorem 1.7, generalizes the approach of Polterovich by expanding it to non-regular distributions. Mangoubi proves, that on $(\mathbb{T}^2 \times M, \sigma \oplus \omega)$ there exists an isotropic singular distribution that satisfies the Hörmander condition, by providing the needed construction. After establishing this, Mangoubi constructs a Riemannian metric on $\mathbb{T}^2 \times M$ by the way which is similar to the "stretching the neck"-type procedure in the approach of Polterovich, and concludes Theorem 1.7 by showing that this is a desired metric. However, the last step of the proof is technically more difficult than the one in the case of Theorem 1.6 of Polterovich. In order to overcome these difficulties, Mangoubi applies the theory of anisotropic Sobolev spaces as developed in [R-S], and the machinery of fractional Sobolev Spaces also known as Bessel Potential Spaces.

In [M], Mangoubi raises the following

Conjecture 1.9. *Let (M, ω) be a closed symplectic manifold of dimension $\geqslant 4$. Then one can find on (M, ω) an isotropic singular distribution that satisfies the Hörmander condition.*

A positive answer to Conjecture 1.9 will imply Conjecture 1.8, as Mangoubi shows in [M]. Interestingly, a negative answer to Conjecture 1.9 would yield to a new type of symplectic rigidity.

In this paper we concentrate on the quasi-Kahler situation. We affirmatively answer to Conjecture 1.8, and prove the following

Theorem 1.10. *Let (M^{2n}, ω) be a closed symplectic manifold of dimension $2n \geqslant 4$. Then there exist Riemannian metrics g on M, compatible with the symplectic structure ω, having arbitrarily large λ_1.*

The proof of Theorem 1.10 relies on the following local result (see below the section describing the notations that we use here):

Proposition 1.11. *For any $R > 0$ and for any $\epsilon > 0$ there exists $\frac{R}{2} < r < R$, and a Riemannian metric g on the domain*

$$D^{2n}_{r,R} = \{x \in \mathbb{R}^{2n} \mid r < |x| < R\},$$

which is compatible with the standard symplectic structure ω_{std} on $D^{2n}_{r,R}$, such that g coincides with the euclidean metric on a neighborhood of the boundary of $D^{2n}_{r,R}$, and such that for any smooth function $f : D^{2n}_{r,R} \to \mathbb{R}$ satisfying

$$\int_{D^{2n}_{r,R}} \|\nabla_g f\|_g^2 \, dg^{2n}_{std} \leqslant 1,$$

there exists some $E \in \mathbb{R}$, such that for any $r < u < R$ we have

$$\int_{S^{2n-1}_u} |f - E|^2 \, dg^{2n-1}_{std} \leqslant \epsilon,$$

where

$$S^{2n-1}_u = \{x \in \mathbb{R}^{2n} \mid |x| = u\},$$

and g_{std} is the euclidean metric on $D^{2n}_{r,R}$.

In our approach, similarly to the approach of Polterovich, and of Mangoubi, we construct the desired Riemannian metric with help of a “stretching the neck”-type procedure. However, in our approach we use ideas that are different from the construction of isotropic distribution that satisfies the Hörmander condition, as it was done by Polterovich [P], and later generalized to the case of singular distributions by Mangoubi [M].

Structure of the paper

In section 2 we sketch an outline of the proof of Theorem 1.10. In section 3 we prove a number of preliminary lemmas, which are used later in the proofs of Proposition 1.11 and Theorem 1.10. The proofs of lemmas from section 3 are quite standard and straightforward, and can be omitted by the reader. In section 4 we prove a local result - Proposition 1.11. Proposition 1.11 is the central ingredient in the proof of Theorem 1.10 in section 5.

Notations

Looking at the euclidean space $\mathbb{R}^d$, by $|\cdot|$ we denote the euclidean norm, and by $\langle\cdot,\cdot\rangle$ we denote the scalar product on $\mathbb{R}^d$. We use the notation g_{std} for the standard euclidean metric on $\mathbb{R}^d$: $g_{std}(u,v) = \langle u,v\rangle$, at each point of $\mathbb{R}^d$. For $r > 0$, we denote by

$$S_r^d = \{x \in \mathbb{R}^d \,|\, |x| = r\}$$

the $(d-1)$-dimensional sphere of radius r centered at the origin, in $\mathbb{R}^d$. For $r > 0$ and $x \in \mathbb{R}^d$, we denote by

$$B_r^d(x) = \{y \in \mathbb{R}^d \,|\, |y - x| < r\}$$

the open ball of radius r centered at x, and by

$$\overline{B}_r^d(x) = \{y \in \mathbb{R}^d \,|\, |y - x| \leqslant r\}$$

the closed ball of radius r centered at x.

On the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$ centered at the origin, consider the spherical Riemannian metric that is induced from the euclidean metric on $\mathbb{R}^{2n}$. For any $x \in S^{2n-1}$ and $\rho > 0$ we denote by $B_\rho^S(x) \subseteq S^{2n-1}$ the ball of radius ρ centered at x, with respect to the spherical Riemannian metric on S^{2n-1}. We will call $B_\rho^S(x)$ as "spherical cap", or as "spherical ball". We denote by B_ρ^S the spherical ρ-ball around the point $(1,0,0,...,0) \in S^{2n-1}$. In the sequel we will also consider hypersurfaces of the form

$$rB_\rho^S(x) = \{ry \,|\, y \in B_\rho^S(x)\} \subseteq S_r^{2n-1},$$

for $r, \rho > 0$, and $x \in S^{2n-1}$. We will call them as spherical caps (spherical balls) as well.

We denote by ω_{std} the standard symplectic form on $\mathbb{R}^{2n}$. For given $0 < r < R < \infty$, we denote the annulus

$$D_{r,R}^{2n} = \{x \in \mathbb{R}^{2n} \,|\, r < |x| < R\}.$$

Given a smooth manifold X^d, and a Riemannian metric g on X, we denote by $\|\cdot\|_g$ the norm on TX induced by g. For a differentiable function $f : X \to \mathbb{R}$, by $\nabla_g f$ we denote the gradient of f with respect to the metric g, so $\nabla_g f(x) \in T_x X$ for any $x \in X$. For $0 \leqslant k \leqslant d$, a k-dimensional submanifold $\Sigma \subseteq X$, and a continuous function $f : \Sigma \to \mathbb{R}$, we denote by $\int_\Sigma f\, dg^k$ the integral of f over Σ with respect to the volume density on Σ which comes from g. We will use the notation $Vol_g(\Sigma)$ for $\int_\Sigma 1\, dg^k$. For a continuous function $h : \Sigma \to \mathbb{R}$, we will say that h is almost equal to some $E \in \mathbb{R}$ in the $L^2(g)$ sense, with respect to the volume density dg^k, if $\int_\Sigma |h-E|^2\, dg^k$ is small; we will say that h is almost equal to some $E \in \mathbb{R}$ in the average $L^2(g)$ sense, with respect to the volume density dg^k, if $\frac{1}{Vol_g(\Sigma)} \int_\Sigma |h-E|^2\, dg^k$ is small. Given two k-dimensional submanifolds $\Sigma_1, \Sigma_2 \subseteq X$ together with a diffeomorphism $\psi : \Sigma_1 \to \Sigma_2$, and continuous functions $h_1 : \Sigma_1 \to \mathbb{R}$, $h_2 : \Sigma_2 \to \mathbb{R}$, we will say that h_2 is close to h_1 in the $L^2(g)$ sense when we identify Σ_2 with Σ_1 via the map ψ, if $\int_{\Sigma_1} |\psi^* h_2 - h_1|^2 dg^k$ is small.

If $\Sigma \subseteq \mathbb{R}^d$ is a k-dimensional submanifold, then by $Vol(\Sigma)$ we mean $Vol_{g_{std}}(\Sigma)$.

For a nice (e.g. open) subset $\Sigma \subseteq S^{2n-1}$, and a continuous function $f : \Sigma \to \mathbb{R}$, we will also use the notation $\int_\Sigma f(\theta)\, d\theta$ for $\int_\Sigma f\, dg_{std}^{2n-1}$.

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2 Outline of the proof

In this section we provide an explanation of the proof of Theorem 1.10 which claims that for a closed symplectic manifold (M^{2n}, ω), there exist Riemannian metrics g on M, compatible with ω, which have arbitrarily large first eigenvalue $\lambda_1(g)$ of the Laplacian associated with g. Recall that one can express $\lambda_1(g)$ as the minimum of

$$\frac{\int_M \|\nabla_g f\|_g^2\, dg^{2n}}{\int_M f^2\, dg^{2n}},$$

when we run over all non-zero functions $f : M \to \mathbb{R}$ having zero mean: $\int_M f dg^{2n} = 0$. Hence the first eigenvalue is large if and only if, for any smooth function $f : M \to \mathbb{R}$ satisfying

$$\int_M f dg^{2n} = 0,$$

$$\int_M \|\nabla_g f\|_g^2 \, dg^{2n} \leqslant 1,$$

we have that f is "almost zero" on M in the $L^2(g)$ sense, or in other words, that $\int_M |f|^2 \, dg^{2n}$ is small.

In the proof of Theorem 1.10, we avoid possible complications with the topology of M, by first proving a local result (Proposition 1.11), and then by passing to any closed symplectic manifold via a smooth triangulation. Below in section 2.2, we briefly explain the proof of Proposition 1.11, and in section 2.3 we briefly explain how we reduce Theorem 1.10 to Proposition 1.11. Section 2.1 explains the "compressing the neck" procedure, that is used in the proofs of Proposition 1.11 and of Theorem 1.10. We direct the reader to section 2.1 first.

2.1 "Compressing the neck" - explanation

We use the following idea (similar constructions were used in [P], [M]). Let (M, ω) be a symplectic manifold (open or closed). Assume that we have fixed a Riemannian metric g_0 on M, which is compatible with the symplectic structure ω, and denote by J_0 the almost complex structure on M, associated with ω and g_0. Let $U \subseteq M$ be an open subset of M, let Y be a smooth non-zero vector field defined on $\overline{U} \subseteq M$, and let $\Sigma \subset \overline{U}$ be a $(2n-1)$-dimensional hypersurface, such that Σ is a proper subset of M. Denote by ψ^t the flow of Y, and assume that for some $T > 0$, we have $\psi^t(\Sigma) \subset U$ for any $t \in (0, T)$. Denote $\Sigma' = \psi^T(\Sigma) \subset \overline{U}$. Given all this setting, we can deform the metric g in the following way: choose a smooth function $b : M \to \mathbb{R}$, such that $b(x) \geqslant 1$ on M, such that $b(x) = 1$ on some open set containing $M \setminus U$, and such that the function $b(\cdot)$ is very large on almost all of U. Then considering the g_0-orthogonal decomposition

$$TU = Span(Y) \oplus Span(J_0 Y) \oplus L,$$

for any $x \in U$ we define

$$g|_x = b(x)^{-1} g_0|_x \oplus b(x) g_0|_x \oplus g_0|_x,$$

and for any $x \notin U$ we set $g|_x = g_0|_x$. Clearly g is compatible with ω as well. By choosing an appropriate function $b(\cdot)$, we can achieve that the hypersurface Σ will become very close to Σ', in metric g, since for any $x \in \Sigma$, the flow trajectory $\{\psi^t(x) \,|\, t \in [0, T]\}$ becomes very short, in metric g. Then, one can easily check that as a consequence, we get the following: for any continuous function $f : \overline{U} \to \mathbb{R}$ which is smooth inside U, and which satisfies

$$\int_U \|\nabla_g f\|_g^2 \, dg^{2n} \leqslant 1,$$

we have that the restriction $f|_{\Sigma'}$ is very close to the restriction $f|_{\Sigma}$, in the $L^2(g_0)$ sense (in fact, in the $L^2(g_1)$ sense, for any initially chosen Riemannian metric g_1 on M), when we identify Σ' with Σ with help of the map ψ^T. This way of passing from the metric g_0 to the metric g reminds the so-called "stretching the neck" procedure, but as we can see, it's purpose is rather to "compress" than to "stretch". Along this section 2, we will call this way as "compressing the neck on U along the vector field Y".

2.2 Local result

Proposition 1.11 tells us, that we can deform the standard euclidean metric on the annulus $D^{2n}_{r,R}$ (for some $\frac{R}{2} < r < R$), such that we will get again a Riemannian metric on $D^{2n}_{r,R}$ that is compatible with the standard symplectic structure ω_{std} on $D^{2n}_{r,R}$, and such that any smooth function on $D^{2n}_{r,R}$ having the L^2-norm of its g-gradient bounded by 1, is almost constant on the cocentric spheres

$$S^{2n-1}_u = \{x \in \mathbb{R}^{2n} \mid |x| = u\},$$

in the $L^2(g_{std})$ sense, where $u \in (r, R)$, and the constant is the same for all $u \in (r, R)$. In our construction, the volume of the annulus $D^{2n}_{r,R}$ is divided into two sub-annuli $D^{2n}_{r,r'}$ and $D^{2n}_{r',R}$ (where $r < r' < R$), when these sub-annuli play different roles in the construction and in the proof. The sub-annulus $D^{2n}_{r',R}$ is chosen to be of width ϵ (i.e. $R - r' = \epsilon$), and the width $r' - r$ of the sub-annulus $D^{2n}_{r,r'}$ is much smaller relative to ϵ. On $D^{2n}_{r',R}$ we choose the metric g to be equal to the standard euclidean metric g_{std}, while on $D^{2n}_{r,r'}$ we construct g by deforming the euclidean metric g_{std} so that the metric g occurs to be "mixed enough" (the precise meaning of this will be clear in the sequel). In the proof that g is the desired metric, the roles of the sub-annuli $D^{2n}_{r',R}$ and $D^{2n}_{r,r'}$ are different. Let us give a rough explanation of this point. Assume that $f : D^{2n}_{r,R} \to \mathbb{R}$ is a smooth function with

$$\int_{D^{2n}_{r,R}} \|\nabla_g f\|^2_g \, dg^{2n}_{std} \leqslant 1. \tag{2.2.1}$$

Then we use the sub-annulus $D^{2n}_{r',R}$ and (2.2.1) to show, that on the sphere $S^{2n-1}_{r'}$ (which is a part of its boundary), there exists a small piece of volume (which in fact is a spherical cap) of size having rate ϵ, such that the restriction of f to it is almost constant in the average $L^2(g_{std})$ sense (Lemma 4.2 in section 4). Then, we use the fact that g is "mixed enough" on $D^{2n}_{r,r'}$, to show that the condition (2.2.1) will imply that in fact, f is almost constant on concentric spheres S^{2n-1}_u for $u \in (r, r']$, in the $L^2(g_{std})$ sense. Then, using the fact that g is standard on $D^{2n}_{r',R}$ and that the width $R-r'$ of $D^{2n}_{r',R}$ is small, we easily show that (2.2.1) implies that f is almost constant on

concentric spheres S_u^{2n-1} in the $L^2(g_{std})$ sense, also for $u \in (r', R)$. To be more precise, we conclude that there exists some $E \in \mathbb{R}$ such that for each $u \in (r, R)$, the restriction of f to S_u^{2n-1} is close to E in the $L^2(g_{std})$ sense, up to $C\epsilon$, where $C = C(n, R)$. Since ϵ can be arbitrary, by replacing ϵ by $\frac{\epsilon}{C}$, we conclude the proposition.

Let us go over the construction of g on $D^{2n}_{r,R}$, and explain the role of $D^{2n}_{r',R}$ and $D^{2n}_{r,r'}$ in the proof in more details.

2.2.1 Sketch of the construction

Consider the sphere $S^{2n-1} \subset \mathbb{R}^{2n}$, and denote by $\tilde{H}$ the Hopf vector field on S^{2n-1}: $\tilde{H}(x) = Jx$ for any $x \in S^{2n-1}$, where J is the standard almost complex structure on $\mathbb{R}^{2n}$. Choose an isometry $\tilde{\alpha} : S^{2n-1} \to S^{2n-1}$ of the sphere, such that the pushforward $\tilde{\alpha}_* \tilde{H}$ of the Hopf vector field $\tilde{H}$, is transverse to the Hopf vector field $\tilde{H}$ at some point $x_1 \in S^{2n-1}$, and hence for some spherical cap $S := B^S_\rho(x_1) \subset S^{2n-1}$ around x_1, the vector field $\tilde{\alpha}_* \tilde{H}$ is transverse to the Hopf vector field $\tilde{H}$ on the closure $\overline{S}$. The radius ρ of the cap S can be chosen to depend only on the dimension $2n-1$. Then we can choose and fix a certain finite collection $\{B^S_{2\epsilon}(x) \,|\, x \in \mathcal{P}\}$ of non-intersecting spherical caps of radius 2ϵ inside S, where $\mathcal{P} \subset S \subset S^{2n-1}$, such that distance from each such $B^S_{2\epsilon}(x)$ (for $x \in \mathcal{P}$) to the boundary ∂S is bounded away from 0, and such that the cardinality $|\mathcal{P}|$ of this collection has rate $\frac{1}{\epsilon^{2n-1}}$. We show (Lemma 4.1 in section 4) that on S^{2n-1} there exists a smooth time dependent vector field $\tilde{Y}^t$, $t \in [0, T]$, such that $\tilde{Y}^t$ is sufficiently C^0-close to the vector field $\tilde{\alpha}_* \tilde{H}$, such that the flow $\tilde{\psi}^t$, $t \in [0, T]$ of $\tilde{Y}^t$ is volume preserving, and such that for any cap in $\{B^S_{2\epsilon}(x) \,|\, x \in \mathcal{P}\}$, there exists a collection of time moments $t_i \in (0, T)$, $i = 1, 2, ..., N$, so that the preimages of this cap under $\tilde{\psi}^{t_i}$, $i = 1, 2, ..., N$, cover the sphere "near-uniformly". Observe that if $\tilde{Y}^t$, $t \in [0, T]$ is sufficiently C^0-close to $\tilde{\alpha}_* \tilde{H}$, then $\tilde{Y}^t$, $t \in [0, T]$ must be transverse to the Hopf vector field $\tilde{H}$ on $\overline{S}$, as well. This observation will be used in the sequel. Now, given this time dependent vector field $\tilde{Y}^t$, $t \in [0, T]$ on S^{2n-1}, for $\delta > 0$ small enough, we set $r' = R - \epsilon$, and $r = r' - T\delta$, and we define a (time independent) vector field Y_δ on $\overline{D^{2n}_{r,r'}}$, which in polar coordinates has the form

$$Y_\delta(r' - \delta t, \theta) = (-\delta, \tilde{Y}^t(\theta)),$$

for $t \in [0, T]$ and $\theta \in S^{2n-1}$. In other words, we obtain the time independent vector field Y_δ on $\overline{D^{2n}_{r,r'}}$ by "spreading" the time dependent vector field $\tilde{Y}^t$, $t \in [0, T]$ through the family of spheres S_u^{2n-1}, $u \in [r, r']$ (and so the radius $u = r' - \delta t$ plays the role of the (reparametrized) time), and then by adding a small component (of amount δ) in the minus-radial direction. Note that as a consequence, if we look at the flow ψ^t_δ of the (time independent) vector field Y_δ, applied to the sphere $S^{2n}_{r'}$, we get just a composition of homotheties of $\mathbb{R}^{2n}$ with the flow $\tilde{\psi}^t$ of the (time dependent) vector

field $\tilde{Y}^t$:

$$\psi^t_\delta(r', \theta) = (r' - \delta t, \tilde{\psi}^t(\theta)).$$

Now we construct the metric g on $D^{2n}_{r,r'}$ by starting with the standard euclidean metric g_{std} on $D^{2n}_{r,r'}$, and then "compressing the neck on $D^{2n}_{r,r'}$ along the vector field Y_δ". On $D^{2n}_{r,R} \setminus D^{2n}_{r,r'}$ we set $g = g_{std}$. Since the vector field $\tilde{Y}^t$, $t \in [0, T]$, is transverse to the Hopf vector field $\tilde{H}$ on $\overline{S}$, we are able to find a certain smooth vector field X_δ (which has a bounded norm, uniformly on δ) on $[r, r'] \cdot \overline{S} \subseteq \overline{D^{2n}_{r,r'}}$, which is orthogonal to $Span(Y_\delta, JY_\delta)$ at every point of $[r, r'] \cdot \overline{S}$, and which radial component equals -1, or in other words, in polar coordinates we have

$$X_\delta(r' - \delta t, \theta) = (-1, \tilde{X}^t_\delta(\theta)),$$

for any $t \in [0, T]$ and $\theta \in \overline{S}$, where $\tilde{X}^t_\delta$, $t \in [0, T]$, is a certain time dependent vector field on $\overline{S} \subset S^{2n-1}$. The flow σ^s_δ of X_δ (which of course, might be defined only partially), satisfies

$$\sigma^s_\delta(r', \theta) = (r' - s, \tilde{\sigma}^s_\delta(\theta)),\ s \in [0, \delta T),$$

where $\tilde{\sigma}^s_\delta$, $s \in [0, \delta T)$ is the flow of the time dependent vector field $\tilde{X}^{\frac{s}{\delta}}_\delta$, $s \in [0, \delta T)$. Note first, that since the time range for the parameter s is small (of length δT), and our vector field X_δ (and hence also $\tilde{X}^t_\delta$) is bounded uniformly on δ, it follows that for δ small enough, the flow $\tilde{\sigma}^s_\delta(\theta)$, $s \in [0, \delta T)$ is well defined when the distance from $\theta \in S$ to the boundary ∂S is bounded away from 0, and moreover the flow $\tilde{\sigma}^s_\delta(\theta)$, $s \in [0, \delta T)$ is arbitrarily C^0-close to the identity when δ is small enough. Secondly, we show that in fact, one can choose X_δ for small $\delta > 0$ as above, such that in addition, the flow $\tilde{\sigma}^s_\delta$, $s \in [0, \delta T)$ is "almost volume preserving" when δ is small enough.

2.2.2 Sketch of the proof

Let $f : D^{2n}_{r,R} \to \mathbb{R}$ be a smooth function such that (2.2.1) holds. First, by looking at the restriction of f to $D^{2n}_{r',R}$, and using (2.2.1), we show (Lemma 4.2 in section 4) that there exists a certain spherical cap $B^S_{2\epsilon}(x_2)$ from our collection of caps (i.e. $x_2 \in \mathcal{P}$), such that on $r' B^S_{2\epsilon}(x_2) \subset S^{2n}_{r'}$, the function f is almost constant (denote this constant by E) in the average $L^2(g_{std})$ sense.

Now for some $s \in (0, \delta T)$, apply the map σ^s_δ (which belongs to the flow of X_δ) to $r' B^S_{2\epsilon}(x_2) \subset S^{2n}_{r'}$. Since δ (and hence s) is small enough, and the vector field X_δ is bounded uniformly on δ, we get that the hypersurface $r' B^S_{2\epsilon}(x_2)$ is very close to the hypersurface $\sigma^s_\delta(r' B^S_{2\epsilon}(x_2)) \subset S^{2n}_{r'-s}$, in metric g. Therefore we can conclude that the restriction of f to $\sigma^s_\delta(r' B^S_{2\epsilon}(x_2))$ is very close to the restriction of f to $r' B^S_{2\epsilon}(x_2)$, in the $L^2(g_{std})$ sense, when we identify $\sigma^s_\delta(r' B^S_{2\epsilon}(x_2))$ with $r' B^S_{2\epsilon}(x_2)$ via the map σ^s_δ (note that here we did not use a "compressing the neck" procedure - it is not necessary since the hypersurfaces $r' B^S_{2\epsilon}(x_2)$ and $\sigma^s_\delta(r' B^S_{2\epsilon}(x_2))$ are already close in the metric g).

Hence we conclude that the restriction of f to $\sigma_\delta^s(r'B_{2\epsilon}^S(x_2))$ is almost equal to E in the average $L^2((\sigma_\delta^s)_* g_{std})$ sense (i.e. when we consider the L^2 norm with respect to the pushforward by the map σ_δ^s, of the standard spherical volume density $dg_{std}^{2n-1}|_{r'B_{2\epsilon}^S(x_2)}$ from $r'B_{2\epsilon}^S(x_2)$ to $\sigma_\delta^s(r'B_{2\epsilon}^S(x_2))$). Now, since $\tilde{\sigma}_\delta^s$ is "almost volume preserving", we conclude that in fact, the restriction of f to $\sigma_\delta^s(r'B_{2\epsilon}^S(x_2))$ is almost equal to E in the average $L^2(g_{std})$ sense. Since for small δ, the map $\tilde{\sigma}_\delta^s(\theta)$ is arbitrarily C^0-close to the identity, we conclude that $\sigma_\delta^s(r'B_{2\epsilon}^S(x_2)) \supseteq (r'-s)B_\epsilon^S(x_2)$, and therefore in particular, on $(r'-s)B_\epsilon^S(x_2)$ the function f is almost equal to the constant E, in the average $L^2(g_{std})$ sense (see Lemma 4.3 in section 4).

We have used the flow of X_δ to show that for each $s \in (0, \delta T)$, the restriction of f to $(r'-s)B_\epsilon^S(x_2)$, is almost equal to the constant E, in the average $L^2(g_{std})$ sense. We can rephrase it by saying that for each $t \in (0,T)$, the restriction of f to $(r'-\delta t)B_\epsilon^S(x_2)$, is almost equal to the constant E, in the average $L^2(g_{std})$ sense. Now let us use the vector field Y_δ, for a similar purpose. Note that we have

$$\psi_\delta^t(S_{r'}^{2n-1}) = S_{r'-\delta t}^{2n-1}.$$

Because we have done the "compressing the neck on $D_{r,r'}^{2n}$ along Y_δ", we can conclude that the restriction of f to $S_{r'-\delta t}^{2n-1}$ is very close to the restriction of f to $S_{r'}^{2n-1}$ in the $L^2(g_{std})$ sense, when we identify $S_{r'-\delta t}^{2n-1}$ with $S_{r'}^{2n-1}$ via the map ψ_δ^t. In particular, the restriction of f to $(r'-\delta t)B_\epsilon^S(x_2) \subset S_{r'-\delta t}^{2n-1}$ is very close to the restriction of f to $(\psi_\delta^t)^{-1}((r'-\delta t)B_\epsilon^S(x_2)) \subset S_{r'}^{2n-1}$, in the $L^2(g_{std})$ sense, when we identify $(r'-\delta t)B_\epsilon^S(x_2)$ with $(\psi_\delta^t)^{-1}((r'-\delta t)B_\epsilon^S(x_2))$ via the map ψ_δ^t. The map $\tilde{\psi}^t$ is volume preserving, and hence the restriction of ψ_δ^t to $S_{r'}^{2n-1}$ is conformally volume preserving, as a map from $S_{r'}^{2n-1}$ to $S_{r'-\delta t}^{2n-1}$. Also recall that the restriction of f to $(r'-\delta t)B_\epsilon^S(x_2)$, is almost equal to the constant E, in the average $L^2(g_{std})$ sense. Hence we can conclude from the described above, that the restriction of f to $(\psi_\delta^t)^{-1}((r'-\delta t)B_\epsilon^S(x_2)) \subset S_{r'}^{2n-1}$, is almost equal to the constant E, in the average $L^2(g_{std})$ sense (see Lemmas 4.4, 4.5).

So we finally conclude that for any $t \in (0,T)$, the restriction of f to $(\psi_\delta^t)^{-1}((r'-\delta t)B_\epsilon^S(x_2)) \subset S_{r'}^{2n-1}$, is almost equal to the constant E, in the average $L^2(g_{std})$ sense. Now, by one of the properties of the flow $\tilde{\psi}^t$ described above, for our point $x_2 \in \mathcal{P}$, there exists a collection of time moments $t_1, t_2, ..., t_N \in (0,T)$, such that the preimages $(\tilde{\psi}^{t_i})^{-1}(B_\epsilon^S(x_2))$, $i = 1, 2, ..., N$, cover the sphere S^{2n-1} "near-uniformly". Clearly this can be rephrased by saying that the preimages $(\psi_\delta^{t_i})^{-1}((r'-\delta t_i)B_\epsilon^S(x_2)) \subset S_{r'}^{2n-1}$, $i = 1, 2, ..., N$, cover the sphere $S_{r'}^{2n-1}$ "near-uniformly". Now, since the restriction of f to each such preimage $(\psi_\delta^{t_i})^{-1}((r'-\delta t_i)B_\epsilon^S(x_2))$, is almost equal to the constant E, in the average $L^2(g_{std})$ sense, we conclude that in fact, the restriction of f to the whole sphere $S_{r'}^{2n-1}$, is almost equal to E, in the $L^2(g_{std})$ sense. Having this in mind, it is already easy to derive the statement of Proposition 1.11. Indeed, if $u \in (r, r')$, then writing $u = r' - \delta t$ for $t \in (0,T)$, we can use the vector field Y_δ once again, identifying $S_{r'}^{2n-1}$ with S_u^{2n-1} with help of the map ψ_δ^t, to conclude that the restriction

of f to the whole sphere S_u^{2n-1}, is almost equal to E, in the $L^2(g_{std})$ sense. Now, if we have $u \in (r, R)$, then we can use the minus-radial vector field $X(x) = -\frac{x}{|x|}$ on $D_{r',R}$, and its flow, identifying the sphere S_u^{2n-1} with the sphere $S_{r'}^{2n-1}$, to conclude that the restriction of f to the whole sphere S_u^{2n-1}, is almost equal to E, in the $L^2(g_{std})$ sense. So finally, for any $u \in (r, R)$, the restriction of f to the whole sphere S_u^{2n-1}, is almost equal to E, in the $L^2(g_{std})$ sense. More precisely, we have shown, that by taking sufficiently small δ, and by appropriately applying the "compressing the neck", we get that for any smooth function $f : D_{r,R}^{2n} \to \mathbb{R}$ satisfying (2.2.1), there exists some $E \in \mathbb{R}$, such that for any $u \in (r, R)$, the restriction of f to S_u^{2n-1}, is close to E in the $L^2(g_{std})$ sense, up to $C\epsilon$, where $C = C(n, R)$. Since we have freedom in the choice of ϵ, we conclude Proposition 1.11.

2.3 From local result to global

Here we briefly describe how we reduce Theorem 1.10 to a local result (Proposition 1.11).

2.3.1 Sketch of the construction

Choose a smooth triangulation of M. Let $\{\Delta_\alpha \,|\, \alpha \in I\}$ be all the open simplices of this triangulation. Choose a Riemannian metric g_0 on M, such that for each $\alpha \in I$, there exists a Darboux neighborhood inside Δ_α, on which g_0 coincides with the euclidean metric.

The desired metric g on M will be constructed by deforming g_0 on a proper subset of Δ_α, for each $\alpha \in I$. For a given $\alpha \in I$, let us describe the way in which we deform g_0 inside Δ_α. For the sake of convenience, we will actually work not on Δ_α, but on the open unit ball $B_1^{2n}(0) \subset \mathbb{R}^{2n}$. In order to make this switch, we use Lemma 3.2 (section 3), which implies that there exists a bi-Lipschitz homeomorphism $\Psi_\alpha : \overline{\Delta_\alpha} \to \overline{B}_1^{2n}(0)$, such that its restriction to Δ_α is a diffeomorphism onto the open unit ball $B_1^d(0)$, and such that its restriction to Δ'_α is a diffeomorphism onto the image, where Δ'_α is the union of Δ_α with all of its open faces. Because of our choice of the metric g_0, WLOG we may assume that the pushforward $\omega_\alpha = (\Psi_\alpha)_*\omega$ of ω from Δ_α to $B_1^{2n}(0)$, and the pushforward $g_{0,\alpha} = (\Psi_\alpha)_* g_0$ of g_0 from Δ'_α to $\Psi_\alpha(\Delta'_\alpha)$, coincide with ω_{std} and g_{std} near the origin $0 \in \overline{B}_1^{2n}(0)$, respectively. Hence we can find some $R_0 > 0$ such that $\omega_\alpha = \omega_{std}$ and $g_{0,\alpha} = g_{std}$ on $B_{R_0}^{2n}(0)$, for each $\alpha \in I$.

Take $0 < R \leqslant R_0$ small enough. By Proposition 1.11, there exists $\frac{R}{2} < r < R$, and a metric g_{loc} on the domain

$$D_{r,R}^{2n} = \{x \in \mathbb{R}^{2n} \,|\, r < |x| < R\},$$

which is compatible with ω_{std}, and is standard near the boundary, such that for any smooth function $f : D^{2n}_{r,R} \to \mathbb{R}$ with

$$\int_{D^{2n}_{r,R}} \|\nabla_g f\|_g^2 \, dg_{std}^{2n} \leqslant 1,$$

there exists some $E \in \mathbb{R}$, such that for any $r < u < R$, the restriction of the function f to S_u^{2n-1}, is very close to the constant function E, in the $L^2(g_{std})$ sense. Denote by $X(x) = -\frac{x}{|x|}$ the "minus-radial vector field" on $\mathbb{R}^{2n} \setminus \{0\}$. Now let us explain how we deform the metric $g_{0,\alpha}$ to a metric g_α, inside Δ_α. At a first step, we define a preliminary metric on $B_1^{2n}(0)$ by starting with the metric $g_{0,\alpha}$ on $B_1^{2n}(0)$, and changing it on $D^{2n}_{r,R}$ to be equal to g_{loc}. Then we define g_α on $B_1^{2n}(0)$ by starting with this preliminary metric on $B_1^{2n}(0)$, and applying the "compressing the neck on $D^{2n}_{R,1}$ along the vector field X".

Finally, we define the metric g on M to be equal to $(\Psi_\alpha)^* g_\alpha$ on each Δ_α, and set $g = g_0$ on $M \setminus (\cup_{\alpha \in I} \Delta_\alpha)$.

2.3.2 Sketch of the proof

Let us show that the metric g will have arbitrarily large λ_1, provided that we took R to be small enough, and did an appropriate "compressing the neck".

Let $f : M \to \mathbb{R}$ be a smooth function with

$$\int_M f \, dg^{2n} = 0,$$

and

$$\int_M \|\nabla_g f\|_g^2 \, dg^{2n} \leqslant 1.$$

Then for any $\alpha \in I$, consider $f_\alpha : \overline{B}_1^{2n}(0) \to \mathbb{R}$ defined as $f_\alpha = (\Psi_\alpha)_* f$ - the pushforward of f by Ψ_α. Then f_α is smooth inside $B_1^{2n}(0)$, and is continuous on $\overline{B}_1^{2n}(0)$. We have

$$\int_{\Delta_\alpha} \|\nabla_g f\|_g^2 \, dg_0^{2n} = \int_{\Delta_\alpha} \|\nabla_g f\|_g^2 \, dg^{2n} \leqslant 1,$$

and hence

$$\int_{B_1^{2n}(0)} \|\nabla_{g_\alpha} f_\alpha\|_{g_\alpha}^2 \, dg_{0,\alpha}^{2n} = \int_{B_1^{2n}(0)} \|\nabla_{g_\alpha} f_\alpha\|_{g_\alpha}^2 \, dg_\alpha^{2n} \leqslant 1. \tag{2.3.2}$$

Now, since $g_\alpha = g_{loc}$ on $D^{2n}_{r,R}$, by Proposition 1.11 we conclude that there exists a constant $E_\alpha \in \mathbb{R}$, such that for any $r < u < R$ (and hence, by continuity, also for $u = r, R$), the restriction of the function f_α to S_u^{2n-1}, is very close to the constant

function E_α, in the $L^2(g_{std})$ sense. Then, because we have done a "compressing the neck" on $D^{2n}_{R,1}$, we get that for any $u \in (R, 1)$, the restriction of f_α to S^{2n-1}_u is very close to the restriction of f_α to S^{2n-1}_R, in the $L^2(g_{std})$ sense, when we identify S^{2n-1}_u with S^{2n-1}_R via a homothety (which is part of the flow of the vector field X). Hence we conclude that also for $u \in (R, 1)$ (and hence, by continuity, also for $u = 1$), the restriction of f_α to S^{2n-1}_u, is very close to the constant function E_α, in the $L^2(g_{std})$ sense. Therefore, integrating over the radius, we conclude that the restriction of f_α to $D^{2n}_{r,1}$, is very close to the constant function E_α, in the $L^2(g_{std})$ sense. Finally, from the fact that the restriction of f_α to S^{2n-1}_r, is very close to the constant function E_α, in the $L^2(g_{std})$ sense, and from (2.3.2), since r is small we conclude that the restriction of f_α to $B^{2n}_r(0)$, is very close to the constant function E_α, in the $L^2(g_{std})$ sense (at this point we use Lemma 3.5 from section 3).

Hence we get the following:
1) The restriction of f_α to $S^{2n-1} = S^{2n-1}_1 = \partial\overline{B}^{2n}_1(0)$, is very close to the constant function E_α, in the $L^2(g_{std})$ sense, and hence in the $L^2(g_{0,\alpha})$ sense.
2) The function f_α is very close to the constant function E_α, in the $L^2(g_{std})$ sense, and hence in the $L^2(g_{0,\alpha})$ sense, on $B^{2n}_1(0)$.

Going back to the manifold M with help of maps Ψ_α, $\alpha \in I$, we get:
1') The restriction of f to $\partial\Delta_\alpha$, is very close to the constant function E_α, in the $L^2(g_0)$ sense.
2') The restriction of f to Δ_α, is very close to the constant function E_α, in the $L^2(g_0)$ sense.

Now we use 1') to conclude that in fact all E_α are close real numbers. Indeed, considering any two adjacent simplices Δ_α and Δ_β having a common face Σ, from 1') we get that the restriction of f to Σ, is very close to both E_α and E_β, in the $L^2(g_0)$ sense. Hence for any two adjacent simplices Δ_α and Δ_β, we have that E_α is close to E_β. Now, since our triangulation is fixed, and since we have a finite number of simplices in our triangulation, we conclude that all E_α, $\alpha \in I$, are close real numbers.

Now fix any $E \in \mathbb{R}$ which is close to all E_α, $\alpha \in I$ (we can take E to be equal to any E_γ). Then from 2') we get that for any $\alpha \in I$, the restriction of f to Δ_α, is very close to E, in the $L^2(g_0)$ sense. But this implies that in fact, f is very close to E on M, in the $L^2(g_0)$ sense, and hence in the $L^2(g)$ sense. Finally, since f is normalized: $\int_M f\, dg^{2n} = 0$, we get that E is very small, and hence we conclude the statement of the theorem.

3 Some preliminary lemmas

Lemma 3.1. *Let X^d be a smooth closed manifold, and let $f : X \to \mathbb{R}$ be a Lipschitz (with respect to some auxiliary Riemannian metric on X) function on X. Then*
1) *There exists a Lipschitz function $F : [0,1] \times X \to \mathbb{R}$ (we consider the standard metric on $[0,1]$), such that $F(1,x) = f(x)$, such that F is smooth on $[0,1) \times X$, and such that we have the following: if $U \subseteq X$ is an open set so that f is smooth on U, then F is smooth on $[0,1] \times U$.*
2) *Moreover, if f was a positive function then F can be chosen to be positive as well.*

Proof of Lemma 3.1. First of all, it is enough to prove 1). Indeed, assume that 1) is true. Now let $f : X \to \mathbb{R}$ be a positive Lipschitz function on X. Then by 1), there exists a function $F : [0,1] \times X \to \mathbb{R}$ with the desired properties. Then for sufficiently small $\epsilon > 0$, replacing F by the function $G : [0,1] \times X \to \mathbb{R}$ defined as $G(t,x) = F(1 - \epsilon + \epsilon t, x)$, we obtain 2).

Now let us show 1). First of all, using partition of unity, we reduce the statement of the lemma to the situation where the support of F lies on a small chart. Hence it is enough to prove the following local statement:
If $f : \mathbb{R}^d \to \mathbb{R}$ is a compactly supported Lipschitz function, with $supp(f) \subset V$ for some open $V \subset \mathbb{R}^d$, then there exists a Lipschitz function $F : [0,1]\times\mathbb{R}^d \to \mathbb{R}$ such that $F(1,x) = f(x)$, such that F is smooth on $[0,1) \times \mathbb{R}^d$, such that $supp(F) \subset [0,1] \times V$, and such that we have the following: if $U \subseteq X$ is an open set such that f is smooth on U, then F is smooth on $[0,1]\times U$. To prove this, we can use a convolution. Indeed, let $f : \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz compactly supported function, with $supp(f) \subset V$ for some open $V \subset \mathbb{R}^d$. Pick a non-negative function $\phi : \mathbb{R}^d \to \mathbb{R}$ having sufficiently small support, and such that $\int_{\mathbb{R}^d} \phi = 1$. Now, define

$$\phi_t(x) = \frac{1}{(1-t)^d}\phi\left(\frac{x}{1-t}\right)$$

for $t \in [0,1)$. Now define $F : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ by $F(1,x) = f(x)$ for $x \in \mathbb{R}^d$, and $F(t,x) = (f * \phi_t)(x)$ for $(t,x) \in [0,1) \times \mathbb{R}^d$. We claim that F is a desired function. Indeed, for $t \in [0,1)$ we have

$$F(t,x) = (f * \phi_t)(x) = \int_{\mathbb{R}^d} \phi_t(y) f(x-y)\,dy = \int_{\mathbb{R}^d} \frac{1}{(1-t)^d}\phi\left(\frac{y}{1-t}\right) f(x-y)\,dy,$$

and after change of variables $z = \frac{y}{1-t}$ we get

$$F(t,x) = \int_{\mathbb{R}^d} \phi(z) f(x - (1-t)z)\,dz, \tag{3.1}$$

Note that the equation (3.1) is true also for $t = 1$, so (3.1) is true for all $(t,x) \in [0,1] \times \mathbb{R}^d$. Therefore we obviously get all the desired properties of F.

$\square$

Lemma 3.2. *Consider an open bounded convex polytope $K \subset \mathbb{R}^d$. Denote by K' the union of K with all of its open faces. Then there exists a bi-Lipschitz homeomorphism $\overline{K} \to \overline{B}_1^d(0)$, such that its restriction to K is a diffeomorphism onto the open unit ball $B_1^d(0)$, and such that its restriction to K' is a diffeomorphism onto the image of K'.*

Proof of Lemma 3.2. WLOG we may assume that $0 \in K$. As a first step, we define the function $F : S^{d-1} \to \mathbb{R}$ as follows: for $\theta \in S^{d-1}$, set $F(\theta) > 0$ to be the unique positive number such that $F(\theta)\theta \in \partial K$. Now define $\tilde{\Psi} : S^{d-1} \to K$ by $\tilde{\Psi}(\theta) = F(\theta)\theta$. Denote $W = \tilde{\Psi}^{-1}(K' \cap \partial K) \subseteq S^{d-1}$ to be preimage by $\tilde{\Psi}$ of the union of all open faces of K. Then clearly, F is smooth on W. Moreover, as it is easy to see, F is Lipschitz on S^{d-1}. Therefore by Lemma 3.1, there exists a positive Lipschitz function $G : [0,1] \times S^{d-1} \to \mathbb{R}$, such that $G(1,\theta) = F(\theta)$, such that G is smooth on $[0,1) \times X$, and such that G is smooth on $[0,1] \times W$. Let $C > 0$ be a Lipschitz constant for G, and let $a = \min_{[0,1]\times S^{d-1}} G$. Choose a smooth function $u : [0,1] \to [0,1]$ such that for some $b, c \in (0,1)$, we have $u(t) = c$ for $t \in [0,b]$, such that $u(1) = 1$ and $u(t) < 1$ for $t < 1$, and such that $|u'(t)| \leqslant \frac{a}{2C}$ on $[0,1]$. Then, given such a function u, choose a smooth function $v : [0,1] \to [0,1]$ such that $v(t) = 1$ for $t \in [b,1]$, such that $v(t) = 0$ for $t \in [0, \frac{b}{2}]$, and such that $v'(t) \geqslant 0$ on $[0,1]$. Now define the function $H : [0,1] \times S^{d-1} \to \mathbb{R}$ as

$$H(t,\theta) = G(u(t),\theta)v(t) + a(1 - v(t)).$$

Then first of all, H is Lipschitz on $[0,1] \times S^{2n-1}$, smooth on $[0,1) \times S^{2n-1}$ and on $[0,1] \times W$, we have $H(1,\theta) = F(\theta)$ for $\theta \in S^{2n-1}$, we have $H(t,\theta) = a$ for $t \in [0, \frac{b}{2}]$ and $\theta \in S^{2n-1}$, and we have $H \geqslant a$ on $[0,1] \times S^{d-1}$. Secondly, for $t \in (0,b]$ we have

$$H_t(t,\theta) = \frac{\partial}{\partial t}(G(c,\theta)v(t) + a(1 - v(t))) = (G(c,\theta) - a)v'(t) \geqslant 0,$$

and for $t \in [b,1)$ we have

$$H_t(t,\theta) = \frac{\partial}{\partial t}G(u(t),\theta) = G_t(u(t),\theta)u'(t) \geqslant -C\frac{a}{2C} = -\frac{a}{2}.$$

Hence in any case, for $t \in (0,1)$ and $\theta \in S^{2n-1}$ we have

$$H_t(t,\theta) \geqslant -\frac{a}{2}.$$

Now define the map $\Psi : \overline{B}_1^d(0) \to \mathbb{R}^d$ so that $\Psi(0) = 0$, and such that on $\overline{B}_1^d(0) \setminus \{0\}$, Ψ is expressed in polar coordinates as

$$\Psi(t,\theta) = (tH(t,\theta),\theta).$$

Then we have $\Psi(x) = ax$ for $x \in B^d_{\frac{b}{2}}(0)$, and hence together with the smoothness of H on $[0,1) \times S^{d-1}$, and the fact that H is Lipschitz on $[0,1] \times S^{2n-1}$, the latter implies that Ψ is a smooth map on $B^d_1(0)$, and that Ψ is Lipschitz on $\overline{B}^d_1(0)$. Also, from the definition of Ψ it follows that it is smooth at the points of $W \subseteq S^{d-1} \subseteq \overline{B}^d_1(0)$. Now, for any $t \in (0,1)$ and $\theta \in S^{d-1}$ we have

$$\frac{\partial}{\partial t}(tH(t,\theta)) = H(t,\theta) + tH_t(t,\theta) \geqslant a - t\frac{a}{2} \geqslant \frac{a}{2}.$$

Together with the fact that $\Psi(x) = ax$ for $x \in B^d_{\frac{b}{2}}(0)$, first of all, the latter implies that Ψ is a bijection from $\overline{B}^d_1(0)$ onto $\overline{K}$, and since Ψ is a continuous map, the compactness of $\overline{B}^d_1(0)$ implies that Ψ is a homeomorphism. Secondly, we get that Ψ is a diffeomorphism from $B^d_1(0)$ onto K, and that the Jacobian of Ψ is bounded from below on $B^d_1(0)$ by a positive constant. That fact, combined with the fact that Ψ is Lipschitz on $B^d_1(0)$, imply that the restriction of Ψ to $B^d_1(0)$ is in fact a bi-Lipschitz map onto K. Hence Ψ is a bi-Lipschitz map from $\overline{B}^d_1(0)$ onto $\overline{K}$.

Therefore $\Psi^{-1} : \overline{K} \to \overline{B}^d_1(0)$ is the desired map.

□

Lemma 3.3. *Consider a d-dimensional open cube $(-1,1)^d \subset \mathbb{R}^d$, endowed with coordinates $(x_1, ..., x_d)$ and with the Riemannian metric g_{std} that comes from the euclidean metric on $\mathbb{R}^d$. Let $\varepsilon > 0$, and let $f : (-1,1)^d \to \mathbb{R}$ be a smooth function, such that*

$$\int_{(-1,1)^d} |\nabla f|^2 \, dg^d_{std} \leqslant \varepsilon.$$

Then there exists some $E \in \mathbb{R}$ such that for any $-1 < x_1 < 1$ we have

$$\int_{\{x_1\}\times(-1,1)^{d-1}} |f - E|^2 dg^{d-1}_{std} \leqslant C\varepsilon,$$

and moreover we have

$$\int_{(-1,1)^d} |f - E|^2 dg^d_{std} \leqslant C\varepsilon,$$

where $C = C(d)$.

Proof of Lemma 3.3. We prove the lemma by induction on the dimension d. First consider $d = 1$. In this case take $E = f(0)$. Then for any $x \in (0,1)$ we have

$$|f(x) - E|^2 = |f(x) - f(0)|^2 = \left|\int_0^x f'(t)\, dt\right|^2 \leqslant x \int_0^x |f'(t)|^2 \, dt \leqslant \int_{-1}^1 |f'(t)|^2 \, dt \leqslant \varepsilon.$$

Similarly we get $|f(x) - E|^2 \leqslant \varepsilon$ for $x \in (-1,0)$. As a consequence, we have

$$\int_{-1}^1 |f(x) - E|^2 \, dx \leqslant 2\varepsilon.$$

This settles the case of $d = 1$.

Now assume that $d \geqslant 2$, and that the lemma is true when the dimension is $d-1$, and let us prove it for the dimension d. Let $f : (-1,1)^d \to \mathbb{R}$ be a smooth function, such that

$$\int_{(-1,1)^d} |\nabla f|^2 \, dg^d_{std} \leqslant \varepsilon.$$

For any $t \in (-1,1)$ define the function $g_t : (-1,1)^{d-1} \to \mathbb{R}$ as

$$g_t(x_2, ..., x_n) = f(t, x_2, x_3, ..., x_n).$$

Then

$$\int_{-1}^{1} \left(\int_{(-1,1)^{d-1}} |\nabla g_t(x_2, ..., x_n)|^2 \, dx_2 \ldots dx_d \right) dt$$
$$\leqslant \int_{(-1,1)^d} |\nabla f|^2 \, dg^d_{std} \leqslant \varepsilon.$$

Hence there exists some $t_1 \in (-1,1)$ such that

$$\int_{(-1,1)^{d-1}} |\nabla g_{t_1}(x_2, ..., x_n)|^2 \, dx_2 \ldots dx_d \leqslant \frac{\varepsilon}{2}.$$

Now, by the induction hypothesis applied to g_{t_1}, there exists $E \in \mathbb{R}$ such that

$$\int_{(-1,1)^{d-1}} |g_{t_1}(x_2, ..., x_n) - E|^2 \, dx_2 \ldots dx_d$$
$$= \int_{(-1,1)^{d-1}} |f(t_1, x_2, ..., x_n) - E|^2 \, dx_2 \ldots dx_d \leqslant \frac{C\varepsilon}{2}.$$

Now let $t \in (-1,1)$ be different from t_1. WLOG assume that $t \in (t_1, 1)$. Then we have

$$\int_{(-1,1)^{d-1}} |f(t, x_2, ..., x_n) - f(t_1, x_2, ..., x_n)|^2 \, dx_2 \ldots dx_d$$
$$= \int_{(-1,1)^{d-1}} \left| \int_{t_1}^{t} \frac{\partial}{\partial x_1} f(x_1, x_2, ..., x_n) \, dx_1 \right|^2 dx_2 \ldots dx_d$$
$$\leqslant \int_{(-1,1)^{d-1}} (t - t_1) \left(\int_{t_1}^{t} \left| \frac{\partial}{\partial x_1} f(x_1, x_2, ..., x_n) \right|^2 dx_1 \right) dx_2 \ldots dx_d$$
$$\leqslant 2 \int_{(-1,1)^{d-1}} \left(\int_{t_1}^{t} \left| \frac{\partial}{\partial x_1} f(x_1, x_2, ..., x_n) \right|^2 dx_1 \right) dx_2 \ldots dx_d$$
$$\leqslant 2 \int_{(-1,1)^{d-1}} \left(\int_{t_1}^{t} |\nabla f(x_1, x_2, ..., x_n)|^2 \, dx_1 \right) dx_2 \ldots dx_d$$
$$\leqslant 2 \int_{(-1,1)^d} |\nabla f(x_1, x_2, ..., x_n)|^2 \, dx_1 \, dx_2 \ldots dx_d \leqslant 2\varepsilon.$$

Hence we conclude that

$$\int_{(-1,1)^{d-1}} |f(t,x_2,...,x_n)-E|^2\,dx_2\ldots dx_d$$

$$\leqslant 2\Big(\int_{(-1,1)^{d-1}} |f(t,x_2,...,x_n)-f(t_1,x_2,...,x_n)|^2\,dx_2\ldots dx_d$$

$$+\int_{(-1,1)^{d-1}} |f(t_1,x_2,...,x_n)-E|^2\,dx_2\ldots dx_d\Big)$$

$$\leqslant 2\left(2\varepsilon+\frac{C\varepsilon}{2}\right)=(C+4)\varepsilon.$$

Similarly, one checks that

$$\int_{(-1,1)^{d-1}} |f(t,x_2,...,x_n)-E|^2\,dx_2\ldots dx_d \leqslant (C+4)\varepsilon$$

also for $t\in(-1,t_1)$. Finally, integrating over t, we get

$$\int_{-1}^{1}\int_{(-1,1)^{d-1}} |f(t,x_2,...,x_n)-E|^2\,dx_2\ldots dx_d\,dt$$

$$=\int_{(-1,1)^d} |f-E|^2 dg^d_{std}$$

$$\leqslant 2(C+4)\varepsilon.$$

□

From Lemmas 3.2 and 3.3 we conclude the following

Corollary 3.4. *Consider the domain $U=(-1,1)\times B_1^{d-1}(0)\subset\mathbb{R}^d$, where*

$$B_1^{d-1}(0)=\{x\in\mathbb{R}^{d-1}\,|\,|x|<1\}\subset\mathbb{R}^{d-1},$$

and the metric g_{std} on U that comes from the euclidean metric on $\mathbb{R}^d$. Let $\varepsilon>0$, and let $f:U\to\mathbb{R}$ be a smooth function, such that

$$\int_U |\nabla f|^2\,dg^d_{std}\leqslant\varepsilon.$$

Then there exists some $E\in\mathbb{R}$, such that for any $-1<x_1<1$ we have

$$\int_{\{x_1\}\times B_1^{d-1}(0)} |f-E|^2 dg^{d-1}_{std}\leqslant C\varepsilon,$$

where $C=C(d)$.

Lemma 3.5. *Let $r, \epsilon > 0$, and let $f : \overline{B}^d_r(0) \to \mathbb{R}$ be a continuous function which is smooth on $B^d_r(0)$, and such that for some $E \in \mathbb{R}$ we have*

$$\int_{S^d_r} |f - E|^2 dg^{d-1}_{std} \leqslant \epsilon,$$

$$\int_{B^d_r(0)} |\nabla f|^2 dg^d_{std} \leqslant 1.$$

Then

$$\int_{B^d_r(0)} |f - E|^2 dg^d_{std} \leqslant C(r^2 + \epsilon r),$$

for some constant $C = C(d)$.

Proof of Lemma 3.5. Define the function $F : \overline{B}^d_1(0) \to \mathbb{R}$ as $F(x) = r^{\frac{d}{2}} f(rx)$, and denote $E' = r^{\frac{d}{2}} E$. Then we have

$$\int_{S^d_1} |F - E'|^2 dg^{d-1}_{std} = r \int_{S^d_r} |f - E|^2 dg^{d-1}_{std} \leqslant r\epsilon,$$

and

$$\int_{B^d_1(0)} |\nabla F|^2 dg^d_{std} = r^2 \int_{B^d_r(0)} |\nabla f|^2 dg^d_{std} \leqslant r^2.$$

Now, by Lemma 3.2, there exists a bi-Lipschitz homeomorphism $[-1,1]^d \to \overline{B}^d_1(0)$, such that its restriction to $(-1,1)^d$ is a diffeomorphism onto the open unit ball $B^d_1(0)$, and such that its restriction to the union of $(-1,1)^d$ with all of open faces of $(-1,1)^d$, is a diffeomorphism onto the image, and let $C = C(d) > 1$ be a bi-Lipschitz constant of this homeomorphism. Denote by $H : [-1,1]^d \to \mathbb{R}$ the pullback of F under this homeomorphism. Then we obtain

$$\int_{\partial[-1,1]^d} |H - E'|^2 dg^{d-1}_{std} \leqslant C^{d-1} r\epsilon, \tag{3.2}$$

and

$$\int_{(-1,1)^d} |\nabla H|^2 dg^d_{std} \leqslant C^{d+2} r^2. \tag{3.3}$$

Now, because of (3.3) and Lemma 3.3, we conclude that there exists some $E'' \in \mathbb{R}$, such that for any $-1 < x_1 < 1$ we have

$$\int_{\{x_1\}\times(-1,1)^{d-1}} |H - E''|^2 dg^{d-1}_{std} \leqslant C' r^2, \tag{3.4}$$

and moreover we have

$$\int_{(-1,1)^d} |H - E''|^2 dg^d_{std} \leqslant C' r^2, \tag{3.5}$$

where $C' = C'(d)$.

Then on one hand, from (3.4) and the uniform continuity of H we get

$$\int_{\{0\}\times(-1,1)^{d-1}} |H - E''|^2 dg_{std}^{d-1} \leqslant C' r^2. \tag{3.6}$$

On the other hand, from (3.2) we get

$$\int_{\{0\}\times(-1,1)^{d-1}} |H - E'|^2 dg_{std}^{d-1} \leqslant C^{d-1} r\epsilon. \tag{3.7}$$

Hence from (3.6), (3.7) we conclude

$$\begin{aligned} 2^{d-1}|E' - E''| &\leqslant 2\Big(\int_{\{0\}\times(-1,1)^{d-1}} |H - E''|^2 dg_{std}^{d-1} + \int_{\{0\}\times(-1,1)^{d-1}} |E' - H|^2 dg_{std}^{d-1}\Big) \\ &\leqslant 2C' r^2 + 2C^{d-1} r\epsilon. \end{aligned}$$

Therefore, by (3.5) we get

$$\begin{aligned} \int_{(-1,1)^d} |H - E'|^2 dg_{std}^d &\leqslant 2\Big(\int_{(-1,1)^d} |H - E''|^2 dg_{std}^d + 2^d |E'' - E'|^2\Big) \\ &\leqslant 2(C' r^2 + 4C' r^2 + 4C^{d-1} r\epsilon) = 10C' r^2 + 8C^{d-1} r\epsilon. \end{aligned}$$

Now, going back to the function F, we conclude

$$\int_{B_1^d(0)} |F - E'|^2 dg_{std}^d \leqslant C^d (10C' r^2 + 8C^{d-1} r\epsilon) \leqslant C''(r^2 + r\epsilon),$$

for $C'' = \max(10C^d C', 8C^{2d-1})$. Therefore we finally get

$$\int_{B_r^d(0)} |f - E|^2 dg_{std}^d \leqslant C''(r^2 + \epsilon r).$$

$\square$

4 Proof of Proposition 1.11

First of all, WLOG, in the proof of this proposition, we may assume that ϵ is small enough.

For some $x \in \mathbb{R}^{2n} \setminus \{0\}$ and a nonzero tangent vector $Y \in T_x(\mathbb{R}^{2n}) \setminus \{0\}$, denote by $Z = \iota(x; Y) \in T_x(\mathbb{R}^{2n})$ the vector that satisfies

$$\langle Z, Y \rangle = \langle Z, JY \rangle = 0,$$

$$\langle Z, X\rangle = 1,$$

and that minimizes the euclidean distance $|Z-X|$, where $X = -\frac{x}{|x|} \in \mathbb{R}^{2n} \cong T_x(\mathbb{R}^{2n})$. It is easy to see that $\iota(x;Y)$ is well defined when $x \notin Span(Y, JY)$, which is equivalent to $Y \notin Span(x, Jx)$, and we will apply ι only in this case. Clearly, $\iota(x;Y)$ depends on x and Y in a smooth way, on its domain of definition.

Restricting to $S^{2n-1} \subset \mathbb{R}^{2n}$, we denote the Hopf vector field by $\tilde{H}(x) = Jx$. Now consider the Hopf vector field $\tilde{H}$ on S^{2n-1}. We can find an isometry $\tilde{\alpha} : S^{2n-1} \to S^{2n-1}$ of the sphere, such that the pushforward $\tilde{\alpha}_* \tilde{H}$ of the Hopf vector field $\tilde{H}$, is transverse to the Hopf vector field $\tilde{H}$ at some point $x_1 \in S^{2n-1}$, and hence for some spherical cap $S = B^S_\rho(x_1) \subset S^{2n-1}$ around x_1, the vector field $\tilde{\alpha}_*\tilde{H}$ is transverse to the Hopf vector field $\tilde{H}$ on the closure $\overline{S}$. Note that the radius ρ of the cap can be chosen to depend only on the dimension $2n-1$. Consider the spherical cap $B^S_{\frac{\rho}{3}}(x_1) \subset S^{2n-1}$, and choose a maximal set of points $\mathcal{P} \subset B^S_{\frac{\rho}{3}}(x_1)$ with the property that the spherical distance between any 2 distinct points of $\mathcal{P}$ is greater or equal to 4ϵ. Since the spherical balls of radius 4ϵ centered at the points of $\mathcal{P}$, cover the ball $B^S_{\frac{\rho}{3}}(x_1)$, we conclude that the cardinality of $\mathcal{P}$ satisfies

$$|\mathcal{P}| \geqslant \frac{Vol(B^S_{\frac{\rho}{3}}(x_1))}{Vol(B^S_{4\epsilon})} \geqslant \frac{c(n)}{\epsilon^{2n-1}},$$

where $Vol(\cdot)$ is evaluated with respect to the volume density g^{2n-1}_{std}.

Lemma 4.1. *There exists some $T > 0$ and a smooth time dependent volume preserving flow $\tilde{\psi}^t$, $t \in [0, T]$ on S^{2n-1}, generated by a time dependent vector field $\tilde{Y}^t$ on S^{2n-1}, such that $\tilde{Y}^t$ is sufficiently C^0-close to the pushforward $\tilde{\alpha}_*\tilde{H}$ of the Hopf vector field on S^{2n-1}, and such that the flow $\tilde{\psi}^t$, $t \in [0, T]$, satisfies the following: Take any $x \in \mathcal{P}$, and denote by $\chi : S^{2n-1} \to \mathbb{R}$ the characteristic function of $B^S_\epsilon(x)$. Then there exist some $t_1, t_2, ..., t_N \in (0, T)$ such that*

$$\frac{1}{N}\sum_{k=1}^{N}(\tilde{\psi}^{t_k})^*\chi \geqslant c(n)\frac{Vol(B^S_\epsilon(x))}{Vol(S^{2n-1})}$$

on S^{2n-1}, where $Vol(\cdot)$ is evaluated with respect to the volume density g^{2n-1}_{std}, and $c(n) > 0$ is some positive constant that depends only on n.

Proof of Lemma 4.1. Along the proof we will use the notation χ_z for the characteristic function $\chi_z : S^{2n-1} \to \mathbb{R}$ of $B^S_\epsilon(z) \subseteq S^{2n-1}$, where $z \in S^{2n-1}$.

Let $\mathcal{Q}$ be a maximal set of points of S^{2n-1} with the property that the spherical distance between any two points of $\mathcal{Q}$ is at least ϵ. Then first of all, spherical balls of radius ϵ centered at points of $\mathcal{Q}$ cover S^{2n-1}. Secondly, spherical balls of radius $\frac{\epsilon}{2}$

centered at the points of $\mathcal{Q}$, do not intersect pairwise, which means that

$$|\mathcal{Q}| \leqslant \frac{Vol(S^{2n-1})}{Vol(B^S_{\frac{\epsilon}{2}})}.$$

Therefore we have

$$\frac{1}{|\mathcal{Q}|}\sum_{y\in\mathcal{Q}}\chi_y \geqslant \frac{1}{|\mathcal{Q}|} \geqslant \frac{Vol(B^S_{\frac{\epsilon}{2}})}{Vol(S^{2n-1})} \geqslant c(n)\frac{Vol(B^S_{\epsilon})}{Vol(S^{2n-1})}$$

on S^{2n-1}.

Now let $x \in \mathcal{P}$ and $y \in \mathcal{Q}$ be any two points. Then there clearly exists a smooth flow $\tilde{\phi}^t_{x,y} : S^{2n-1} \to S^{2n-1}$, $t \in [0,1]$, consisting of isometries of S^{2n-1}, such that $\tilde{\phi}^t_{x,y}$ is identity when t is sufficiently close to 0 or 1, and such that

$$(\tilde{\phi}^{\frac{1}{2}}_{x,y})^*\chi_x = \chi_y.$$

Denote by $\tilde{\xi}^t : S^{2n-1} \to S^{2n-1}$ the flow of $\tilde{\alpha}_*\tilde{H}$ - this is also a flow of isometries of S^{2n-1}, and we have that $\tilde{\xi}^{2\pi}$ is the identity diffeomorphism of S^{2n-1}. Now take $M_{x,y} \in \mathbb{N}$ to be sufficiently large, and define the flow $\tilde{\psi}^t_{x,y} : S^{2n-1} \to S^{2n-1}$, $t \in [0, 4M_{x,y}\pi]$, as

$$\tilde{\psi}^t_{x,y} = \tilde{\xi}^t \circ \tilde{\phi}^{\frac{t}{4M_{x,y}\pi}}_{x,y}.$$

If we take $M_{x,y}$ to be sufficiently large, then the vector field that generates the flow $\tilde{\psi}^t_{x,y}$ will be sufficiently C^0-close to $\tilde{\alpha}_*\tilde{H}$. In addition, we have that $\tilde{\psi}^t_{x,y}$ equals to $\tilde{\xi}^t$ when t is close to the endpoints 0 and $4M_{x,y}\pi$, so that in particular $\tilde{\psi}^{4M_{x,y}\pi}_{x,y}$ is the identity diffeomorphism of S^{2n-1}, and also we have that

$$(\tilde{\psi}^{2M_{x,y}\pi}_{x,y})^*\chi_x = \chi_y.$$

Now define the flow $\tilde{\psi}^t$, $t \in [0, T]$ to be the concatenation of flows $\tilde{\psi}^t_{x,y}$, when we run over all $x \in \mathcal{P}$ and $y \in \mathcal{Q}$. We claim that $\tilde{\psi}^t$ is a desired flow. Indeed, first of all it is smooth since $\tilde{\psi}^t_{x,y}$ equals to $\tilde{\xi}^t$ when t is close to the endpoints 0 and $4M_{x,y}\pi$, for every $x \in \mathcal{P}$ and $y \in \mathcal{Q}$. Secondly, the vector field that generates $\tilde{\psi}^t$, is sufficiently C^0-close to $\tilde{\alpha}_*\tilde{H}$. Fixing any $x \in \mathcal{P}$, we have that for any $y \in \mathcal{Q}$ there exists $t_y \in (0, T)$ such that $(\tilde{\psi}^{t_y})^*\chi_x = \chi_y$. Therefore we have

$$\frac{1}{|\mathcal{Q}|}\sum_{y\in\mathcal{Q}}(\tilde{\psi}^{t_y})^*\chi_x = \frac{1}{|\mathcal{Q}|}\sum_{y\in\mathcal{Q}}\chi_y \geqslant c(n)\frac{Vol(B^S_{\epsilon})}{Vol(S^{2n-1})} = c(n)\frac{Vol(B^S_{\epsilon}(x))}{Vol(S^{2n-1})}$$

on S^{2n-1}. Finally, the flow $\tilde{\psi}^t$ consists of isometries of S^{2n-1}, and hence is volume preserving.

□

Consider the time dependent vector field $\tilde{Y}^t$ and its flow $\tilde{\psi}^t$ on S^{2n-1}, guaranteed by Lemma 4.1. If in Lemma 4.1 the vector field $\tilde{Y}^t$ is sufficiently C^0-close to $\tilde{\alpha}_*\tilde{H}$, then $\tilde{Y}^t$ will also be transverse to the Hopf vector field $\tilde{H}$ on the closure $\overline{S}$. Choose a sufficiently small $\delta > 0$, and denote

$$r' = R - \epsilon,$$

$$r = r' - T\delta = R - \epsilon - T\delta.$$

Clearly, if ϵ and δ are small enough, then we will have $r > \frac{R}{2}$. Define the (time independent) vector field Y_δ on $\overline{D^{2n}_{r,r'}}$, which in polar coordinates has the form

$$Y_\delta(r' - \delta t, \theta) = (-\delta, \tilde{Y}^t(\theta)),$$

where $t \in [0, T]$ and $\theta \in S^{2n-1}$. Define the vector field

$$X_\delta(x) = \iota(x; Y_\delta(x)),$$

on $x \in [r, r'] \cdot \overline{S} = [r, r'] \cdot \overline{B^S_\rho(x_1)}$ (for small δ, $\iota(x; Y_\delta(x))$ is well defined on $x \in [r, r'] \cdot \overline{S} = [r, r'] \cdot \overline{B^S_\rho(x_1)}$). Consider a smooth function $a : \mathbb{R} \to \mathbb{R}$, such that $a(t) \geqslant 1$ for any $t \in \mathbb{R}$, such that $a(t) = 1$ for any $t \notin (\frac{\delta^2}{2}, T - \frac{\delta^2}{2})$, and such that $a(t)$ is large enough on $[\delta^2, T - \delta^2]$. Now define $b : D^{2n}_{r,r'} \to \mathbb{R}$ as

$$b(x) = a\left(\frac{r' - |x|}{\delta}\right).$$

Let us give the definition of the metric g on $D^{2n}_{r,R}$. On $D^{2n}_{r,R} \setminus D^{2n}_{r,r'}$ we set $g = g_{std}$. Now consider $D^{2n}_{r,r'}$. Looking at g_{std}-orthogonal decomposition

$$TD^{2n}_{r,r'} = Span(Y_\delta) \oplus Span(JY_\delta) \oplus L,$$

we define

$$g|_x = b(x)^{-1} g_{std}|_x \oplus b(x) g_{std}|_x \oplus g_{std}|_x$$

for any $x \in D^{2n}_{r,R}$.

Our main statement is the following:

Claim: If we pick sufficiently small δ, and then choose the function $a(\cdot)$ to be large enough on $[\delta^2, T - \delta^2]$, then the constructed metric g will satisfy the following:
For any smooth function $f : D^{2n}_{r,R} \to \mathbb{R}$ satisfying

$$\int_{D^{2n}_{r,R}} \|\nabla_g f\|_g^2 \, dg^{2n}_{std} \leqslant 1,$$

there exists some $E \in \mathbb{R}$, such that for any $r < u < R$ we have

$$\int_{S^{2n-1}_u} |f - E|^2 \, dg^{2n-1}_{std} \leqslant C\epsilon,$$

where $C = C(n, R)$ depends only on n and R.

The rest of the proof of Proposition 1.11 will be devoted for proving this claim. In the sequel we will assume that we have chosen δ to be small enough, and the function $a(\cdot)$ to be sufficiently large on $[\delta^2, T - \delta^2]$.

Note first, that since X_δ is orthogonal to $Span(Y_\delta, JY_\delta)$ on $(r, r'] \cdot \overline{S} = (r, r'] \cdot \overline{B^S_\rho(x_1)}$, it follows that

$$\|X_\delta\|_g = |X_\delta|$$

on $(r, r'] \cdot \overline{S} = (r, r'] \cdot \overline{B^S_\rho(x_1)}$.

Denote by ψ^t_δ the flow of the vector field Y_δ, and by σ^s_δ the flow of the vector field X_δ. In polar coordinates we have

$$\psi^t_\delta(r', \theta) = (r' - \delta t, \tilde{\psi}^t(\theta)),$$

for $t \in [0, T]$. Also, in polar coordinates the vector field X_δ can be written as

$$X_\delta(r' - \delta t, \theta) = (-1, \tilde{X}^t_\delta(\theta)),$$

for $t \in [0, T]$, where $\tilde{X}^t_\delta(\theta)$ is a time-dependent vector field. Note that $\tilde{X}^t_\delta(\theta)$ is well defined for $\theta \in \overline{S} = \overline{B^S_\rho(x_1)}$, $t \in [0, T]$, when δ is small enough. For the flow σ^s_δ of X_δ we have

$$\sigma^s_\delta(r', \theta) = (r' - s, \tilde{\sigma}^s_\delta(\theta)),$$

for $s \in [0, \delta T]$, where $\tilde{\sigma}^s_\delta(\theta)$ is a flow on S^{2n-1} (defined only partially) which is generated by the vector field $\tilde{X}^{\frac{s}{\delta}}_\delta(\theta)$, when $s \in [0, \delta T]$.

Based on Corollary 3.4 (section 3), we are able to prove the following

Lemma 4.2. *Let $f : D^{2n}_{r,R} \to \mathbb{R}$ be a smooth function, satisfying*

$$\int_{D^{2n}_{r,R}} \|\nabla_g f\|^2_g \, dg^{2n}_{std} \leqslant 1.$$

Then there exists a point $x_2 \in \mathcal{P}$ such that for some $E \in \mathbb{R}$ we have

$$\frac{1}{Vol(r' B^S_{2\epsilon}(x_2))} \int_{r' B^S_{2\epsilon}(x_2)} |f - E|^2 \, dg^{2n-1}_{std} \leqslant C\epsilon,$$

where $C = C(n, R)$.

Proof of Lemma 4.2. For $x \in S^{2n-1}$ denote $U_{x,\epsilon} = (r', R) \cdot B^S_{2\epsilon}(x) \subseteq D^{2n}_{r',R}$. Note that in polar coordinates $U_{x,\epsilon}$ is

$$U_{x,\epsilon} = (R - \epsilon, R) \times B^S_{2\epsilon}(x) \subset (0, \infty) \times S^{2n-1}.$$

Define the following domain in $\mathbb{R}^{2n}$:

$$U_\epsilon = (-\epsilon,\epsilon)\times B^{2n-1}_\epsilon(0) = (-\epsilon,\epsilon)\times\{y\in\mathbb{R}^{2n-1}\,|\,|y|<\epsilon\}\subset\mathbb{R}^{2n}.$$

It is easy to see that for small enough ϵ, for any $x\in S^{2n-1}$ there exists a diffeomorphism $\Psi_x : U_{x,\epsilon}\to U_\epsilon$, such that we have

$$\frac{1}{C'}g_{std}\leqslant \Psi_x^* g_{std}\leqslant C' g_{std},$$

where $C'=C'(n,R)$, and such that in polar coordinates the map Ψ_x has the form

$$\Psi_x(u,\theta)=\left(2\left(R-\frac{\epsilon}{2}-u\right),\tilde{\Psi}_x(\theta)\right),$$

for some diffeomorphism $\tilde{\Psi}_x : B^S_{2\epsilon}(x)\to B^{2n-1}_\epsilon(0)$.

Now, since the distance between any two distinct points of $\mathcal{P}$ is greater or equal to 4ϵ, it follows that all $U_{x,\epsilon}$, for $x\in\mathcal{P}$, pairwise do not intersect. Therefore we conclude that

$$\sum_{x\in\mathcal{P}}\int_{U_{x,\epsilon}}|\nabla f|^2\,dg^{2n}_{std}\leqslant\int_{D^{2n}_{r',R}}|\nabla f|^2\,dg^{2n}_{std}=\int_{D^{2n}_{r',R}}\|\nabla_g f\|^2_g\,dg^{2n}_{std}\leqslant\int_{D^{2n}_{r,R}}\|\nabla_g f\|^2_g\,dg^{2n}_{std}\leqslant 1.$$

Keeping in mind that $|\mathcal{P}|\geqslant\frac{c(n)}{\epsilon^{2n-1}}$, we conclude that there exists some $x_2\in\mathcal{P}$ such that

$$\int_{U_{x_2,\epsilon}}|\nabla f|^2\,dg^{2n}_{std}\leqslant\frac{1}{|\mathcal{P}|}\leqslant C\epsilon^{2n-1},$$

where $C=C(n)$. Now look at $\Psi:=\Psi_{x_2}:U_{x_2,\epsilon}\to U_\epsilon$. Define the function $h:U_\epsilon\to\mathbb{R}$ as $h=f\circ\Psi^{-1}$. Then we have

$$\int_{U_\epsilon}|\nabla h|^2\,dg^{2n}_{std}\leqslant CC'^{2n+2}\epsilon^{2n-1}.$$

Applying a rescaling, we define the function $H:U\to\mathbb{R}$, where $U=(-1,1)\times B^{d-1}_1(0)\subset\mathbb{R}^d$, as $H(x)=h(\epsilon x)$. Then $|\nabla H(x)|^2=\epsilon^2|\nabla h(\epsilon x)|^2$, and the Jacobian of the map $x\mapsto\epsilon x$ is ϵ^{2n}, hence we get

$$\epsilon^{2n-2}\int_U|\nabla H|^2\,dg^{2n}_{std}\leqslant CC'^{2n+2}\epsilon^{2n-1},$$

or

$$\int_U|\nabla H|^2\,dg^{2n}_{std}\leqslant CC'^{2n+2}\epsilon. \tag{4.1}$$

Now, applying Corollary 3.4 (section 3) to (4.1), we conclude that there exists $E\in\mathbb{R}$ such that

$$\int_{\{x_1\}\times B^{2n-1}_1(0)}|H-E|^2dg^{2n-1}_{std}\leqslant C''\epsilon$$

for any $x_1 \in (-1,1)$, where $C'' = C''(n,R)$. Rescaling back, we get

$$\int_{\{x_1\}\times B^{2n-1}_{\epsilon}(0)} |h-E|^2 dg^{2n-1}_{std} \leqslant C''\epsilon^{2n}$$

for any $x_1 \in (-1,1)$. Going back to $f = h\circ\Psi$, we get that for any $u \in (r',R)$ we have

$$\int_{uB^S_{2\epsilon}(x_2)} |f-E|^2\, dg^{2n-1}_{std} \leqslant C'^{2n-1}C''\epsilon^{2n}.$$

Hence by continuity we have

$$\int_{r'B^S_{2\epsilon}(x_2)} |f-E|^2\, dg^{2n-1}_{std} \leqslant C'^{2n-1}C''\epsilon^{2n}.$$

Finally, keeping in mind that $Vol(r'B^S_{2\epsilon}(x_2)) \geqslant c'\epsilon^{2n-1}$ for $c' = c'(n,R)$, we conclude the statement of the lemma. □

Lemma 4.3. *Let f and $x_2 \in \mathcal{P}$ be as in Lemma 4.2. Denote $B^t_\epsilon = (r'-\delta t)B^S_\epsilon(x_2)$, for $t \in [0,T)$. Then provided that δ is small enough, we will have*

$$\frac{1}{Vol(B^t_\epsilon)}\int_{B^t_\epsilon} |f-E|^2\, dg^{2n-1}_{std} \leqslant C\epsilon,$$

for all $t \in (0,T)$. (In this lemma the constant $C = C(n,R)$ might be different from the one in Lemma 4.2).

Proof of Lemma 4.3. Since ϵ is small, then for any $x \in \mathcal{P}$ we have $B^S_{2\epsilon}(x) \subseteq B^S_{\frac{\rho}{2}}(x_1)$, and in particular, $B^S_{2\epsilon}(x_2) \subseteq B^S_{\frac{\rho}{2}}(x_1)$. Look at the vector field

$$X_\delta(r'-\delta t,\theta) = (-1, \tilde{X}^t_\delta(\theta)).$$

The flow of X_δ satisfies

$$\sigma^s_\delta(r',\theta) = (r'-s, \tilde{\sigma}^s_\delta(\theta)),$$

for $s \in [0,\delta T]$. The flow $\tilde{\sigma}^s_\delta$ is generated by the vector field $\tilde{X}^{\frac{s}{\delta}}_\delta$, when $s \in [0,\delta T]$. Let us make a time rescaling, concentrating on the time parameter $t = \frac{s}{\delta}$. Define the flow $\tilde{\varsigma}^t_\delta$ on a part of S^{2n-1} as $\tilde{\varsigma}^t_\delta(\theta) = \tilde{\sigma}^{\delta t}_\delta(\theta)$, when $t \in [0,T]$. Then the flow $\tilde{\varsigma}^t_\delta$ is generated by the vector field $\delta\tilde{X}^t_\delta$. Up till now we had a family of vector fields $\delta \to \tilde{X}^t_\delta$ on $\overline{S} \subseteq S^{2n-1}$, defined for small $\delta > 0$. However, it is quite easy to see that we can extend this family also to $\delta = 0$ in a natural way. Indeed, we have that

$$\tilde{X}^t_\delta(\theta) = \theta + \iota((r'-\delta t,\theta); Y_\delta(r'-\delta t,\theta)) = \theta + \iota((r'-\delta t)\theta; -\delta\theta + (r'-\delta t)\tilde{Y}^t(\theta)).$$

Hence if we define

$$\tilde{X}^t_0(\theta) = \theta + \iota(r'\theta; r'\tilde{Y}^t(\theta)),$$

then $\tilde{X}^t_\delta(\theta)$ is well defined for small $\delta \geqslant 0$, for $t \in [0,T]$ and for $\theta \in \overline{S}$, and depends on δ, t, and θ, in a smooth way.

So we get that the flow $\tilde{\zeta}^t_\delta$ is generated by the vector field $\delta \tilde{X}^t_\delta$, and that the family of vector fields $\tilde{X}^t_\delta(\theta)$, depends on small $\delta \geqslant 0$, on $\theta \in \overline{S}$, and on $t \in [0,T]$, in a smooth way. From here we can conclude the following:
1) Since in addition $\tilde{X}^t_\delta(\theta)$ is well defined for $\theta \in \overline{S}$ and $t \in [0,T]$, then for δ small enough, the flow $\tilde{\zeta}^t_\delta(\theta)$ is well defined for $\theta \in B^S_{\frac{\rho}{2}}(x_1)$ and $t \in [0,T]$, and we have $\tilde{\zeta}^t_\delta(\theta) \in S = B^S_\rho(x_1)$ for any $\theta \in B^S_{\frac{\rho}{2}}(x_1)$ and $t \in [0,T]$.
2) Moreover, if δ is small enough, then for any $x \in B^S_{\frac{\rho}{3}}(x_1)$ we will have $\tilde{\zeta}^t_\delta(B^S_{2\epsilon}(x)) \supseteq B^S_\epsilon(x)$ for all $t \in [0,T]$.
3) Finally, for any $t \in [0,T]$, the Jacobian of the map $B^S_{\frac{\rho}{2}}(x_1) \to S^{2n-1}$ given by $\theta \mapsto \tilde{\zeta}^t_\delta(\theta)$, can be arbitrarily close to 1, uniformly on $\theta \in B^S_{\frac{\rho}{2}}(x_1)$ and $t \in [0,T]$. In particular, if δ is small enough, then the Jacobian lies between $\frac{1}{2}$ and 2.

Now assume that δ is small enough so that 1), 2), 3) above are satisfied. Then, translating these properties to the flow of $\tilde{\sigma}^s_\delta(\theta)$, we get:
1') The flow $\tilde{\sigma}^s_\delta(\theta)$ is well defined for $\theta \in B^S_{\frac{\rho}{2}}(x_1)$ and $s \in [0,\delta T]$, and we have $\tilde{\sigma}^s_\delta(\theta) \in S = B^S_\rho(x_1)$ for any $\theta \in B^S_{\frac{\rho}{2}}(x_1)$ and $s \in [0,\delta T]$.
2') For any $x \in B^S_{\frac{\rho}{3}}(x_1)$ we have $\tilde{\sigma}^s_\delta(B^S_{2\epsilon}(x)) \supseteq B^S_\epsilon(x)$ for all $s \in [0,\delta T]$.
3') For any $s \in [0,\delta T]$, the Jacobian of the map $B^S_{\frac{\rho}{2}}(x_1) \to S^{2n-1}$ given by $\theta \mapsto \tilde{\sigma}^s_\delta(\theta)$, lies between $\frac{1}{2}$ and 2, at any $\theta \in B^S_{\frac{\rho}{2}}(x_1)$.

We are now ready to prove our lemma. Define the function $F : [0,\delta T) \times B^S_{2\epsilon}(x_2) \to \mathbb{R}$ as

$$F(s,\theta) = f(\sigma^s_\delta(r',\theta)) = f(r' - s, \tilde{\sigma}^s_\delta(\theta)).$$

For any $0 < s_1 < \delta T$ we have

$$\int_{B^S_{2\epsilon}(x_2)} |f(r' - s_1, \tilde{\sigma}^{s_1}_\delta(\theta)) - f(r',\theta)|^2 \, d\theta = \int_{B^S_{2\epsilon}(x_2)} |F(s_1,\theta) - F(0,\theta)|^2 \, d\theta$$

$$= \int_{B^S_{2\epsilon}(x_2)} \left| \int_0^{s_1} \frac{\partial}{\partial s} F(s,\theta) \, ds \right|^2 d\theta \leqslant \int_{B^S_{2\epsilon}(x_2)} s_1 \int_0^{s_1} \left| \frac{\partial}{\partial s} F(s,\theta) \right|^2 ds \, d\theta$$

$$= s_1 \int_{B^S_{2\epsilon}(x_2)} \int_0^{s_1} \left| L_{X_\delta} f(\sigma^s_\delta(r',\theta)) \right|^2 \, ds \, d\theta$$

$$\leqslant \delta T \int_{B^S_{2\epsilon}(x_2)} \int_0^{s_1} \|\nabla_g f(\sigma^s_\delta(r',\theta))\|^2_g \cdot \|X_\delta(\sigma^s_\delta(r',\theta))\|^2_g \, ds \, d\theta$$

$$= \delta T \int_{B^S_{2\epsilon}(x_2)} \int_0^{s_1} \|\nabla_g f(\sigma^s_\delta(r',\theta))\|^2_g \cdot |X_\delta(\sigma^s_\delta(r',\theta))|^2 \, ds \, d\theta.$$

We have

$$X_\delta(u,\theta) = \iota((u,\theta); Y_\delta(u,\theta)) = \iota(u\theta; -\delta\theta + u\tilde{Y}^t(\theta)),$$

for $u \in [r, r']$ and $\theta \in \overline{S} = \overline{B^S_\rho(x_1)}$. Therefore, if $\tilde{Y}^t$ is sufficiently C^0-close to $\tilde{\alpha}_* \tilde{H}$, and if ϵ, δ are small enough, then because of continuous dependence of $\iota(\cdot\,;\cdot)$ on its arguments, we can conclude that $X_\delta(u,\theta)$ is C^0-close to $\iota(R\theta; R\tilde{\alpha}_*\tilde{H}(\theta)) = \iota(\theta; \tilde{\alpha}_*\tilde{H}(\theta))$, and hence in this case we have $|X_\delta(u,\theta)|^2 \leqslant C'$ for any $u \in [r, r']$, $\theta \in \overline{S} = \overline{B^S_\rho(x_1)}$, and small $\delta > 0$, where $C' = C'(n)$. Hence returning to our chain of estimates, we get

$$\begin{aligned}
&\int_{B^S_{2\epsilon}(x_2)} |f(r' - s_1, \tilde{\sigma}^{s_1}_\delta(\theta)) - f(r', \theta)|^2 \, d\theta \\
&\leqslant \delta T \int_{B^S_{2\epsilon}(x_2)} \int_0^{s_1} \|\nabla_g f(\sigma^s_\delta(r',\theta))\|^2_g \cdot |X_\delta(\sigma^s_\delta(r',\theta))|^2 \, ds \, d\theta \\
&\leqslant \delta T C' \int_{B^S_{2\epsilon}(x_2)} \int_0^{s_1} \|\nabla_g f(\sigma^s_\delta(r',\theta))\|^2_g \, ds \, d\theta
\end{aligned}$$

Now, because of 3'), the Jacobian of the map $\Phi : (0, \delta T) \times B^S_{2\epsilon}(x_2) \to D_{r,r'}$ given by $(s, \theta) \mapsto \sigma^s_\delta(r', \theta) = (r' - s, \tilde{\sigma}^s_\delta(\theta))$, is greater or equal to $\frac{(r'-s)^{2n-1}}{2}$, which is greater than $\frac{r^{2n-1}}{2}$, which in turn, is greater than $\frac{\left(\frac{R}{2}\right)^{2n-1}}{2} = \frac{R^{2n-1}}{2^{2n}}$, for small ϵ and δ. Hence returning to our chain of estimates, we get

$$\begin{aligned}
&\int_{B^S_{2\epsilon}(x_2)} |f(r' - s_1, \tilde{\sigma}^{s_1}_\delta(\theta)) - f(r', \theta)|^2 \, d\theta \\
&\leqslant \delta T C' \int_{B^S_{2\epsilon}(x_2)} \int_0^{s_1} \|\nabla_g f(\sigma^s_\delta(r',\theta))\|^2_g \, ds \, d\theta \\
&\leqslant \delta \frac{2^{2n} T C'}{R^{2n-1}} \int_{\Phi([0,\delta T) \times B^S_{2\epsilon}(x_2))} \|\nabla_g f\|^2_g \, dg^{2n}_{std} \\
&\leqslant \delta \frac{2^{2n} T C'}{R^{2n-1}} \int_{D^{2n}_{r,R}} \|\nabla_g f\|^2_g \, dg^{2n}_{std} \leqslant \delta \frac{2^{2n} T C'}{R^{2n-1}}.
\end{aligned} \tag{4.2}$$

Now, since by Lemma 4.2 we have

$$\frac{1}{Vol(r' B^S_{2\epsilon}(x_2))} \int_{r' B^S_{2\epsilon}(x_2)} |f - E|^2 \, dg^{2n-1}_{std} \leqslant C\epsilon,$$

we get

$$\frac{1}{Vol(B^S_{2\epsilon}(x_2))} \int_{B^S_{2\epsilon}(x_2)} |f(r', \theta) - E|^2 \, d\theta \leqslant C\epsilon. \tag{4.3}$$

Hence from (4.2) and (4.3) we conclude that

$$\begin{aligned}
&\frac{1}{Vol(B^S_{2\epsilon}(x_2))} \int_{B^S_{2\epsilon}(x_2)} |f(r' - s_1, \tilde{\sigma}^{s_1}_\delta(\theta)) - E|^2 \, d\theta \\
&\leqslant 2\Bigg(\frac{1}{Vol(B^S_{2\epsilon}(x_2))} \int_{B^S_{2\epsilon}(x_2)} |f(r' - s_1, \tilde{\sigma}^{s_1}_\delta(\theta)) - f(r', \theta)|^2 \, d\theta
\end{aligned}$$

$$+\frac{1}{Vol(B^S_{2\epsilon}(x_2))}\int_{B^S_{2\epsilon}(x_2)}|f(r',\theta)-E|^2\,d\theta\Bigg)\leqslant 2\left(\delta\frac{2^{2n}TC'}{R^{2n-1}Vol(B^S_{2\epsilon}(x_2))}+C\epsilon\right)\leqslant 3C\epsilon,$$

where the latter inequality is true if δ is small enough. Now, from 3') we know that the Jacobian of the map $\tilde{\sigma}^{s_1}_{\delta}(\theta):B^S_{\frac{\rho}{2}}(x_1)\to S^{2n-1}$ is not greater than 2, hence we conclude

$$\frac{1}{Vol(B^S_{2\epsilon}(x_2))}\int_{\tilde{\sigma}^{s_1}_{\delta}(B^S_{2\epsilon}(x_2))}|f(r'-s_1,\theta)-E|^2\,d\theta$$

$$\leqslant\frac{2}{Vol(B^S_{2\epsilon}(x_2))}\int_{B^S_{2\epsilon}(x_2)}|f(r'-s_1,\tilde{\sigma}^{s_1}_{\delta}(\theta))-E|^2\,d\theta\leqslant 6C\epsilon.$$

Because of 2') we have $\tilde{\sigma}^{s_1}_{\delta}(B^S_{2\epsilon}(x_2))\supseteq B^S_{\epsilon}(x_2)$, so we get

$$\frac{1}{Vol(B^S_{2\epsilon}(x_2))}\int_{B^S_{\epsilon}(x_2)}|f(r'-s_1,\theta)-E|^2\,d\theta$$

$$\leqslant\frac{1}{Vol(B^S_{2\epsilon}(x_2))}\int_{\tilde{\sigma}^{s_1}_{\delta}(B^S_{2\epsilon}(x_2))}|f(r'-s_1,\theta)-E|^2\,d\theta\leqslant 6C\epsilon.$$

Therefore we finally obtain

$$\frac{1}{Vol(B^S_{\epsilon}(x_2))}\int_{B^S_{\epsilon}(x_2)}|f(r'-s_1,\theta)-E|^2\,d\theta$$

$$=\frac{Vol(B^S_{2\epsilon}(x_2))}{Vol(B^S_{\epsilon}(x_2))}\frac{1}{Vol(B^S_{2\epsilon}(x_2))}\int_{B^S_{\epsilon}(x_2)}|f(r'-s_1,\theta)-E|^2\,d\theta$$

$$\leqslant 6C\frac{Vol(B^S_{2\epsilon}(x_2))}{Vol(B^S_{\epsilon}(x_2))}\epsilon\leqslant C''\epsilon,$$

for some $C''=C''(n,R)$. The latter means that for any $t\in(0,T)$, for $B^t_{\epsilon}=(r'-\delta t)B^S_{\epsilon}(x_2)$ we have

$$\frac{1}{Vol(B^t_{\epsilon})}\int_{B^t_{\epsilon}}|f-E|^2\,dg^{2n-1}_{std}\leqslant C''\epsilon,$$

$\square$

Lemma 4.4. *Let $f:D^{2n}_{r,R}\to\mathbb{R}$ be a smooth function satisfying*

$$\int_{D^{2n}_{r,R}}\|\nabla_g f\|^2_g\,dg^{2n}_{std}\leqslant 1.$$

Then for any $0\leqslant t_1<t_2<T$ we have

$$\int_{S^{2n-1}}|f(r'-\delta t_2,\tilde{\psi}^{t_2}(\theta))-f(r'-\delta t_1,\tilde{\psi}^{t_1}(\theta))|^2\,d\theta\leqslant C\delta.$$

(In this lemma the constant $C=C(n,R)$ might be different from those in lemmas 4.2, 4.3).

Proof of Lemma 4.4. Define the function $F : [0,T) \times S^{2n-1} \to \mathbb{R}$ as

$$F(t,\theta) = f(\psi_\delta^t(r',\theta)) = f(r' - \delta t, \tilde{\psi}^t(\theta)).$$

Then for any $0 \leqslant t_1 < t_2 < T$ we have

$$\int_{S^{2n-1}} |f(r' - \delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r' - \delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta = \int_{S^{2n-1}} |F(t_2,\theta) - F(t_1,\theta)|^2 \, d\theta$$

$$= \int_{S^{2n-1}} \left| \int_{t_1}^{t_2} \frac{\partial}{\partial t} F(t,\theta) \, dt \right|^2 d\theta \leqslant \int_{S^{2n-1}} (t_2 - t_1) \int_{t_1}^{t_2} \left| \frac{\partial}{\partial t} F(t,\theta) \right|^2 dt \, d\theta$$

$$= (t_2 - t_1) \int_{S^{2n-1}} \int_{t_1}^{t_2} \left| L_{Y_\delta} f(\psi_\delta^t(r',\theta)) \right|^2 dt \, d\theta$$

$$\leqslant (t_2 - t_1) \int_{S^{2n-1}} \int_{t_1}^{t_2} \|\nabla_g f(\psi_\delta^t(r',\theta))\|_g^2 \cdot \|Y_\delta(\psi_\delta^t(r',\theta))\|_g^2 \, dt \, d\theta$$

$$= (t_2 - t_1) \int_{S^{2n-1}} \int_{t_1}^{t_2} \|\nabla_g f(\psi_\delta^t(r',\theta))\|_g^2 \cdot |Y_\delta(\psi_\delta^t(r',\theta))|^2 \cdot b(\psi_\delta^t(r',\theta))^{-2} \, dt \, d\theta$$

$$= (t_2 - t_1) \int_{S^{2n-1}} \int_{t_1}^{t_2} \|\nabla_g f(\psi_\delta^t(r',\theta))\|_g^2 \cdot |Y_\delta(\psi_\delta^t(r',\theta))|^2 \cdot b(r' - \delta t, \tilde{\psi}^t(\theta))^{-2} \, dt \, d\theta$$

$$= (t_2 - t_1) \int_{S^{2n-1}} \int_{t_1}^{t_2} \|\nabla_g f(\psi_\delta^t(r',\theta))\|_g^2 \cdot |Y_\delta(\psi_\delta^t(r',\theta))|^2 \cdot a(t)^{-2} \, dt \, d\theta$$

$$\leqslant \frac{t_2 - t_1}{\left(\min_{t \in [t_1,t_2]} a(t)\right)^2} \int_{S^{2n-1}} \int_{t_1}^{t_2} \|\nabla_g f(\psi_\delta^t(r',\theta))\|_g^2 \cdot |Y_\delta(\psi_\delta^t(r',\theta))|^2 \, dt \, d\theta \, .$$

We have

$$|Y_\delta(u,\theta)|^2 = |-\delta\theta + u\tilde{Y}^t(\theta)|^2 = \delta^2 + u^2 |\tilde{Y}^t(\theta)|^2,$$

for $u \in [r,r']$ and $\theta \in S^{2n-1}$. According to the property that $\tilde{Y}^t$ is sufficiently C^0-close to $\tilde{\alpha}_* \tilde{H}$ (Lemma 4.1), the norm $|\tilde{Y}^t(\theta)|$ is sufficiently close to 1, so we may assume that $|\tilde{Y}^t(\theta)| \leqslant 2$ for all $t \in [0,T]$ and $\theta \in S^{2n-1}$. Hence we have

$$|Y_\delta(u,\theta)|^2 = \delta^2 + u^2 |\tilde{Y}^t(\theta)|^2 \leqslant \delta^2 + 4u^2 \leqslant \delta^2 + 4r'^2,$$

for $u \in [r,r']$ and $\theta \in S^{2n-1}$. Therefore, returning to our chain of estimates, we have

$$\int_{S^{2n-1}} |f(r' - \delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r' - \delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta$$

$$\leqslant \frac{t_2 - t_1}{\left(\min_{t \in [t_1,t_2]} a(t)\right)^2} \int_{S^{2n-1}} \int_{t_1}^{t_2} \|\nabla_g f(\psi_\delta^t(r',\theta))\|_g^2 \cdot |Y_\delta(\psi_\delta^t(r',\theta))|^2 \, dt \, d\theta$$

$$\leqslant \frac{(t_2 - t_1)(\delta^2 + 4r'^2)}{\left(\min_{t \in [t_1,t_2]} a(t)\right)^2} \int_{S^{2n-1}} \int_{t_1}^{t_2} \|\nabla_g f(\psi_\delta^t(r',\theta))\|_g^2 \, dt \, d\theta \, .$$

The Jacobian of the map $(0,T)\times S^{2n-1}\to D^{2n}_{r,r'}$ given by

$$(t,\theta)\mapsto \psi^t_\delta(r',\theta)=(r'-\delta t,\tilde{\psi}^t(\theta)),$$

equals to $\delta(r'-\delta t)^{2n-1}$, since the flow $\tilde{\psi}^t$ on S^{2n-1} is volume preserving. Hence this Jacobian is greater than $\delta(r'-\delta T)^{2n-1}=\delta r^{2n-1}$ at every point of $(0,T)\times S^{2n-1}$. So returning again to our chain of estimates, we conclude that

$$\begin{aligned}
&\int_{S^{2n-1}} |f(r'-\delta t_2,\tilde{\psi}^{t_2}(\theta))-f(r'-\delta t_1,\tilde{\psi}^{t_1}(\theta))|^2\,d\theta\\
&\leqslant \frac{(t_2-t_1)(\delta^2+4r'^2)}{\left(\min_{t\in[t_1,t_2]}a(t)\right)^2}\int_{S^{2n-1}}\int_{t_1}^{t_2}\|\nabla_g f(\psi^t_\delta(r',\theta))\|_g^2\,dt\,d\theta\\
&\leqslant \frac{t_2-t_1}{\delta\left(\min_{t\in[t_1,t_2]}a(t)\right)^2}\cdot\frac{\delta^2+4r'^2}{r^{2n-1}}\cdot\int_{D^{2n}_{r,r'}}\|\nabla_g f\|_g^2\,dg^{2n}_{std}\\
&\leqslant \frac{t_2-t_1}{\delta\left(\min_{t\in[t_1,t_2]}a(t)\right)^2}\cdot\frac{\delta^2+4r'^2}{r^{2n-1}}\cdot\int_{D^{2n}_{r,R}}\|\nabla_g f\|_g^2\,dg^{2n}_{std}\\
&\leqslant \frac{t_2-t_1}{\delta\left(\min_{t\in[t_1,t_2]}a(t)\right)^2}\cdot\frac{\delta^2+4r'^2}{r^{2n-1}}.
\end{aligned}$$

Now, provided that ϵ,δ are small enough, we have

$$\frac{\delta^2+4r'^2}{r^{2n-1}}<\frac{5R^2}{(\frac{1}{2}R)^{2n-1}}=:C.$$

So we conclude that

$$\int_{S^{2n-1}} |f(r'-\delta t_2,\tilde{\psi}^{t_2}(\theta))-f(r'-\delta t_1,\tilde{\psi}^{t_1}(\theta))|^2\,d\theta\leqslant C\frac{t_2-t_1}{\delta\left(\min_{t\in[t_1,t_2]}a(t)\right)^2},$$

for any $0\leqslant t_1<t_2<T$. In particular, for $0\leqslant t_1<t_2\leqslant\delta^2$ we get

$$\begin{aligned}
\int_{S^{2n-1}} |f(r'-\delta t_2,\tilde{\psi}^{t_2}(\theta))-f(r'-\delta t_1,\tilde{\psi}^{t_1}(\theta))|^2\,d\theta &\leqslant C\frac{t_2-t_1}{\delta\left(\min_{t\in[t_1,t_2]}a(t)\right)^2}\\
&\leqslant C\frac{\delta^2}{\delta}=C\delta.
\end{aligned}$$

Analogously, for any $T-\delta^2\leqslant t_1<t_2<T$ we have

$$\begin{aligned}
\int_{S^{2n-1}} |f(r'-\delta t_2,\tilde{\psi}^{t_2}(\theta))-f(r'-\delta t_1,\tilde{\psi}^{t_1}(\theta))|^2\,d\theta &\leqslant C\frac{t_2-t_1}{\delta\left(\min_{t\in[t_1,t_2]}a(t)\right)^2}\\
&\leqslant C\frac{\delta^2}{\delta}=C\delta.
\end{aligned}$$

Finally, for $\delta^2 \leqslant t_1 < t_2 \leqslant T - \delta^2$ we have

$$\int_{S^{2n-1}} |f(r' - \delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r' - \delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta \leqslant C \frac{t_2 - t_1}{\delta \left(\min_{t \in [t_1, t_2]} a(t) \right)^2}$$

$$\leqslant C \frac{T}{\delta \left(\min_{t \in [t_1, t_2]} a(t) \right)^2}.$$

If we choose the function $a(\cdot)$ to be sufficiently large on $[\delta^2, T - \delta^2]$ (it is enough to require $a(t) \geqslant \frac{\sqrt{T}}{\delta}$ for $t \in [\delta^2, T - \delta^2]$), then we will get

$$\int_{S^{2n-1}} |f(r' - \delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r' - \delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta$$

$$\leqslant C \frac{T}{\delta \left(\min_{t \in [t_1, t_2]} a(t) \right)^2} \leqslant C\delta.$$

These three cases, combined together, imply that for any $0 \leqslant t_1 < t_2 < T$ we have

$$\int_{S^{2n-1}} |f(r' - \delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r' - \delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta \leqslant 9C\delta.$$

□

Lemma 4.5. *Let f, and $x_2 \in \mathcal{P}$, and B_ϵ^t, $t \in [0, T)$ be as in Lemma 4.3. Then for any $t \in (0, T)$, looking at the preimage $(\psi_\delta^t)^{-1}(B_\epsilon^t) \subset S_{r'}^{2n-1}$, we have*

$$\frac{1}{Vol((\psi_\delta^t)^{-1}(B_\epsilon^t))} \int_{(\psi_\delta^t)^{-1}(B_\epsilon^t)} |f - E|^2 \, dg_{std}^{2n-1} \leqslant C\epsilon.$$

(In this lemma the constant $C = C(n, R)$ might be different from those in lemmas 4.2, 4.3, 4.4).

Proof of Lemma 4.5. By Lemma 4.3, we have

$$\frac{1}{Vol(B_\epsilon^t)} \int_{B_\epsilon^t} |f - E|^2 \, dg_{std}^{2n-1} \leqslant C\epsilon,$$

which means

$$\frac{1}{Vol(B_\epsilon^S(x_2))} \int_{B_\epsilon^S(x_2)} |f(r' - \delta t, \theta) - E|^2 \, d\theta \leqslant C\epsilon, \tag{4.4}$$

for all $t \in (0, T)$. By Lemma 4.4, for any $0 \leqslant t_1 < t_2 < T$ we have

$$\int_{S^{2n-1}} |f(r' - \delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r' - \delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta \leqslant C\delta,$$

so in particular taking some $t \in (0, T)$ and considering $t_1 = 0$, $t_2 = t$, we get

$$\int_{S^{2n-1}} |f(r' - \delta t, \tilde{\psi}^t(\theta)) - f(r', \theta)|^2 \, d\theta \leqslant C\delta,$$

which implies that

$$\frac{1}{Vol(B^S_\epsilon(x_2))}\int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))} |f(r'-\delta t,\tilde{\psi}^t(\theta))-f(r',\theta)|^2\,d\theta \leqslant \frac{C\delta}{Vol(B^S_\epsilon(x_2))}.$$

Also, since the flow $\tilde{\psi}^t : S^{2n-1}\to S^{2n-1}$ is volume preserving, it follows from (4.4) that

$$\begin{aligned}&\frac{1}{Vol(B^S_\epsilon(x_2))}\int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))} |f(r'-\delta t,\tilde{\psi}^t(\theta))-E|^2\,d\theta\\ &=\frac{1}{Vol(B^S_\epsilon(x_2))}\int_{B^S_\epsilon(x_2)} |f(r'-\delta t,\theta)-E|^2\,d\theta \leqslant C\epsilon.\end{aligned}$$

Hence we conclude

$$\begin{aligned}&\frac{1}{Vol(B^S_\epsilon(x_2))}\int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))} |f(r',\theta)-E|^2\,d\theta\\ &\leqslant 2\Bigg(\frac{1}{Vol(B^S_\epsilon(x_2))}\int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))} |f(r'-\delta t,\tilde{\psi}^t(\theta))-E|^2\,d\theta\\ &+\frac{1}{Vol(B^S_\epsilon(x_2))}\int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))} |f(r'-\delta t,\tilde{\psi}^t(\theta))-f(r',\theta)|^2\,d\theta\Bigg)\\ &\leqslant 2\left(C\epsilon+\frac{C\delta}{Vol(B^S_\epsilon(x_2))}\right).\end{aligned}$$

If δ is small enough, then we will have

$$2\left(C\epsilon+\frac{C\delta}{Vol(B^S_\epsilon(x_2))}\right)\leqslant 3C\epsilon.$$

Therefore we conclude that

$$\frac{1}{Vol(B^S_\epsilon(x_2))}\int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))} |f(r',\theta)-E|^2\,d\theta \leqslant 3C\epsilon,$$

or in other words,

$$\frac{1}{Vol((\psi^t_\delta)^{-1}(B^t_\epsilon))}\int_{(\psi^t_\delta)^{-1}(B^t_\epsilon)} |f-E|^2\,dg^{2n-1}_{std}\leqslant 3C\epsilon.$$

□

Let us finally conclude the claim stated above. Let g be the metric on $D^{2n}_{r,R}$ defined as above, and assume that δ is small enough and that the function $a(\cdot)$ is large enough on $[\delta^2, T-\delta^2]$. Let $f : D^{2n}_{r,R}\to\mathbb{R}$ be a smooth function satisfying

$$\int_{D^{2n}_{r,R}} \|\nabla_g f\|^2_g\,dg^{2n}_{std}\leqslant 1.$$

By Lemma 4.2, there exists a point $x_2 \in \mathcal{P}$ and some $E \in \mathbb{R}$ such that

$$\frac{1}{Vol(r'B^S_{2\epsilon}(x_2))} \int_{r'B^S_{2\epsilon}(x_2)} |f - E|^2 \, dg^{2n-1}_{std} \leqslant C\epsilon.$$

Then, by Lemma 4.5, for any $t \in (0, T)$, looking at the preimage $(\psi^t_\delta)^{-1}(B^t_\epsilon) \subseteq S^{2n-1}_{r'}$ of $B^t_\epsilon = (r' - \delta t)B^S_\epsilon(x_2)$, we have

$$\frac{1}{Vol((\psi^t_\delta)^{-1}(B^t_\epsilon))} \int_{(\psi^t_\delta)^{-1}(B^t_\epsilon)} |f - E|^2 \, dg^{2n-1}_{std} \leqslant C'\epsilon,$$

where $C' = C'(n, R)$. But since

$$\begin{aligned}
&\frac{1}{Vol((\psi^t_\delta)^{-1}(B^t_\epsilon))} \int_{(\psi^t_\delta)^{-1}(B^t_\epsilon)} |f - E|^2 \, dg^{2n-1}_{std} \\
&= \frac{1}{Vol((\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2)))} \int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))} |f(r', \theta) - E|^2 \, d\theta \\
&= \frac{1}{Vol(B^S_\epsilon)} \int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))} |f(r', \theta) - E|^2 \, d\theta \\
&= \int_{S^{2n-1}} \left(\frac{1}{Vol(B^S_\epsilon)} (\tilde{\psi}^t)^* \chi(\theta) \right) |f(r', \theta) - E|^2 \, d\theta,
\end{aligned}$$

we get

$$\int_{S^{2n-1}} \left(\frac{1}{Vol(B^S_\epsilon)} (\tilde{\psi}^t)^* \chi(\theta) \right) |f(r', \theta) - E|^2 \, d\theta \leqslant C'\epsilon, \tag{4.5}$$

where $\chi : S^{2n-1} \to \mathbb{R}$ is the characteristic function of $B^S_\epsilon(x_2)$. Now, by Lemma 4.1, there exist some $t_1, t_2, ..., t_N \in (0, T)$ such that

$$\frac{1}{N} \sum_{k=1}^{N} (\tilde{\psi}^{t_k})^* \chi \geqslant c \frac{Vol(B^S_\epsilon(x_2))}{Vol(S^{2n-1})} \tag{4.6}$$

on S^{2n-1}. Averaging (4.5) over $t = t_1, ..., t_N$, and using (4.6), we get

$$\begin{aligned}
&\frac{c}{Vol(S^{2n-1})} \int_{S^{2n-1}} |f(r', \theta) - E|^2 \, d\theta \\
&= \int_{S^{2n-1}} \frac{1}{Vol(B^S_\epsilon)} \frac{c \cdot Vol(B^S_\epsilon(x_2))}{Vol(S^{2n-1})} |f(r', \theta) - E|^2 \, d\theta \\
&\leqslant \int_{S^{2n-1}} \left(\frac{1}{Vol(B^S_\epsilon)} \frac{1}{N} \sum_{k=1}^{N} (\tilde{\psi}^{t_k})^* \chi(\theta) \right) |f(r', \theta) - E|^2 \, d\theta \leqslant C'\epsilon,
\end{aligned}$$

and hence

$$\int_{S^{2n-1}} |f(r', \theta) - E|^2 \, d\theta \leqslant C''\epsilon, \tag{4.7}$$

where

$$C'' = \frac{C' Vol(S^{2n-1})}{c}.$$

Now, fixing any $t \in (0, T)$, and applying Lemma 4.4 for $t_1 = 0$, $t_2 = t$, we get

$$\int_{S^{2n-1}} |f(r' - \delta t, \tilde{\psi}^t(\theta)) - f(r', \theta)|^2 \, d\theta \leqslant C''' \delta,$$

which together with (4.7) give us

$$\int_{S^{2n-1}} |f(r' - \delta t, \theta) - E|^2 \, d\theta = \int_{S^{2n-1}} |f(r' - \delta t, \tilde{\psi}^t(\theta)) - E|^2 \, d\theta$$
$$\leqslant 2 \left(\int_{S^{2n-1}} |f(r' - \delta t, \tilde{\psi}^t(\theta)) - f(r', \theta)|^2 \, d\theta + \int_{S^{2n-1}} |f(r', \theta) - E|^2 \, d\theta \right)$$
$$\leqslant 2(C''' \delta + C'' \epsilon) \leqslant 3C'' \epsilon,$$

since $\tilde{\psi}^t$ is volume preserving, and δ is small enough. Thus we have proved that

$$\int_{S^{2n-1}} |f(u, \theta) - E|^2 \, d\theta \leqslant 3C'' \epsilon, \tag{4.8}$$

for any $u \in (r, r')$. Now consider the case when $u \in (r', R)$. Define the vector field X on $\mathbb{R}^{2n} \setminus \{0\}$, as $X(x) = -\frac{x}{|x|}$ for $x \in \mathbb{R}^{2n} \setminus \{0\}$. Then keeping in mind that $g = g_{std}$ on $D^{2n}_{r',R}$, we obtain

$$\int_{S^{2n-1}} |f(u, \theta) - f(r', \theta)|^2 \, d\theta$$
$$= \int_{S^{2n-1}} \left| \int_{r'}^{u} \frac{\partial}{\partial s} f(s, \theta) \, ds \right|^2 d\theta \leqslant \int_{S^{2n-1}} (u - r') \int_{r'}^{u} \left| \frac{\partial}{\partial s} f(s, \theta) \right|^2 ds \, d\theta$$
$$= (u - r') \int_{S^{2n-1}} \int_{r'}^{u} |L_X f(s, \theta)|^2 \, ds \, d\theta$$
$$\leqslant (u - r') \int_{S^{2n-1}} \int_{r'}^{u} \|\nabla_g f(s, \theta)\|_g^2 \cdot \|X(s, \theta)\|_g^2 \, ds \, d\theta$$
$$= (u - r') \int_{S^{2n-1}} \int_{r'}^{u} \|\nabla_g f(s, \theta)\|_g^2 \, ds \, d\theta.$$

The Jacobian of the map $(r', R) \times S^{2n-1} \to D^{2n}_{r',R}$ given by

$$(s, \theta) \mapsto s\theta,$$

equals to s^{2n-1}, and hence is greater than r'^{2n-1} at every point of $(r', R) \times S^{2n-1}$. So returning again to our chain of estimates, we conclude that

$$\int_{S^{2n-1}} |f(u, \theta) - f(r', \theta)|^2 \, d\theta \leqslant (u - r') \int_{S^{2n-1}} \int_{r'}^{u} \|\nabla_g f(s, \theta)\|_g^2 \, ds \, d\theta$$

$$\leqslant \frac{u-r'}{r'^{2n-1}}\int_{D^{2n}_{r',u}} \|\nabla_g f\|_g^2 \, dg^{2n}_{std} \leqslant \frac{u-r'}{r'^{2n-1}}\int_{D^{2n}_{r,R}} \|\nabla_g f\|_g^2 \, dg^{2n}_{std} \leqslant \frac{u-r'}{r'^{2n-1}}.$$

But we have $u - r' < R - r' = \epsilon$, and for small ϵ, δ we have $r'^{2n-1} \geqslant \left(\frac{R}{2}\right)^{2n-1}$, hence we get

$$\int_{S^{2n-1}} |f(u,\theta) - f(r',\theta)|^2 \, d\theta \leqslant \left(\frac{2}{R}\right)^{2n-1} \epsilon. \tag{4.9}$$

Therefore, from (4.7) and (4.9) we get

$$\begin{aligned} &\int_{S^{2n-1}} |f(u,\theta) - E|^2 \, d\theta \\ \leqslant 2 \left(\int_{S^{2n-1}} |f(r',\theta) - E|^2 \, d\theta + \int_{S^{2n-1}} |f(u,\theta) - f(r',\theta)|^2 \, d\theta \right) \\ \leqslant 2 \left(C''\epsilon + \left(\frac{2}{R}\right)^{2n-1} \epsilon \right) = \left(2C'' + \frac{2^{2n}}{R^{2n-1}} \right) \epsilon, \end{aligned} \tag{4.10}$$

for any $u \in (r', R)$. Combining (4.7), (4.8), and (4.10), we conclude that for any $u \in (r, R)$ we have

$$\int_{S^{2n-1}} |f(u,\theta) - E|^2 \, d\theta \leqslant \left(3C'' + \frac{2^{2n}}{R^{2n-1}} \right) \epsilon.$$

Hence for any $u \in (r, R)$ we have

$$\int_{S^{2n-1}_u} |f - E|^2 \, dg^{2n-1}_{std} = u^{2n-1} \int_{S^{2n-1}} |f(u,\theta) - E|^2 \, d\theta$$

$$\leqslant R^{2n-1} \left(3C'' + \frac{2^{2n}}{R^{2n-1}} \right) \epsilon = (3C''R^{2n-1} + 2^{2n})\epsilon.$$

This finishes the proof of our claim, and hence of the proposition.

5 Proof of Theorem 1.10

Choose a smooth triangulation of M, and let $\Delta_\alpha \subseteq M$, $\alpha \in I$, be the open simplices of this triangulation. Choose a Riemannian metric g_0 on M, such that for each $\alpha \in I$ there exists a Darboux neighborhood inside Δ_α, on which g_0 coincides with the euclidean metric.

For $\alpha \in I$, denote by Δ'_α the union of Δ_α with all of its open faces. Then for each $\alpha \in I$, by Lemma 3.2, there exists a bi-Lipschitz homeomorphism $\Psi_\alpha : \overline{\Delta_\alpha} \to \overline{B}^{2n}_1(0)$, such that Ψ_α is a diffeomorphism from Δ_α onto $B^{2n}_1(0)$, and also is a diffeomorphism from Δ'_α onto the image. Because of our choice of g_0, WLOG we may assume that the pushforward ω_α of the symplectic structure ω from $B^{2n}_1(0)$ to Δ_α, equals to ω_{std} (i.e.

is standard) near the origin, and that the pushforward $g_{0,\alpha}$ of the metric g_0 from Δ'_α to its image $\Psi_\alpha(\Delta'_\alpha) \subseteq \overline{B}_1^{2n}(0)$, coincides with the standard euclidean metric g_{std} near the origin. Hence we can find some $0 < R_0 < 1$, such that $\omega_\alpha = \omega_{std}$ and $g_{0,\alpha} = g_{std}$ on $B^{2n}_{R_0}(0)$, for all $\alpha \in I$. Let $C > 0$ be a bi-Lipschitz constant for all Ψ_α, $\alpha \in I$, when we consider the metric g_0 on $\overline{\Delta_\alpha}$, and the metric g_{std} on $\overline{B}_1^{2n}(0)$. Then we get

$$\frac{1}{C} g_{std} \leqslant g_{0,\alpha} \leqslant C g_{std}$$

on $\Psi_\alpha(\Delta'_\alpha)$, for each $\alpha \in I$.

Now pick any $0 < R \leqslant R_0$. After choosing R, pick a small enough $\epsilon > 0$. Then by Proposition 1.11, there exists $\frac{R}{2} < r < R$, and a metric g_{loc} on the domain

$$D^{2n}_{r,R} = \{x \in \mathbb{R}^{2n} \,|\, r < |x| < R\},$$

having all the desired properties. Consider the "minus-radial vector field" $X(x) = -\frac{x}{|x|}$ on $D^{2n}_{R,1}$. Choose a sufficiently small $\delta' > 0$, and choose a smooth function $\hat{a} : \mathbb{R} \to \mathbb{R}$, such that $\hat{a}(u) = 1$ for $u \notin (R + \frac{\delta'}{2}, 1 - \frac{\delta'}{2})$, such that $\hat{a}(u) \geqslant 1$ for all $u \in \mathbb{R}$, and such that $\hat{a}(u)$ is sufficiently large on $[R+\delta', 1-\delta']$. Define $\hat{b} : B^{2n}_1(0) \to \mathbb{R}$ as $\hat{b}(x) = \hat{a}(|x|)$. Denote by $J_{0,\alpha}$ the almost complex structure that relates ω_α and $g_{0,\alpha}$. Now we define the metric g_α on $B^{2n}_1(0)$ as follows: on $D^{2n}_{r,R}$ we set $g_\alpha = g_{loc}$; on $D^{2n}_{R,1}$, looking at the $g_{0,\alpha}$-orthogonal decomposition

$$TD^{2n}_{R,1} = Span(X) \oplus Span(J_{0,\alpha}X) \oplus L,$$

we define

$$g_\alpha|_x = \hat{b}(x)^{-1} g_{0,\alpha}|_x \oplus \hat{b}(x) g_{0,\alpha}|_x \oplus g_{0,\alpha}|_x$$

at each $x \in D^{2n}_{R,1}$; finally, on $B^{2n}_1(0) \setminus (D^{2n}_{r,R} \cup D^{2n}_{R,1})$ we set $g_\alpha = g_{0,\alpha} = g_{std}$. Clearly, g_α is a smooth Riemannian metric on $B^{2n}_1(0)$, which is compatible with ω_α, and which coincides with $g_{0,\alpha}$ near the boundary of $B^{2n}_1(0)$. Also, exactly as in the case of Lemma 4.4, but now using the flow of the vector field X, one can prove the following

Claim

If the function $\hat{a}(\cdot)$ is sufficiently large on $[R+\delta', 1-\delta']$, then for any smooth function $h : D^{2n}_{R,1} \to \mathbb{R}$ satisfying

$$\int_{D^{2n}_{R,1}} \|\nabla_{g_\alpha} h\|^2_{g_\alpha} \, dg^{2n}_{std} \leqslant 1,$$

we have the following: for any $R < u_1 < u_2 < 1$,

$$\int_{S^{2n-1}} |h(u_2, \theta) - h(u_1, \theta)|^2 \, d\theta \leqslant C'\delta',$$

where $C' = C'(n, R)$.

Finally, we define the metric g on M as follows: for any $\alpha \in I$, on Δ_α we set $g = \Psi_\alpha^* g_\alpha$; on $M \setminus (\cup_{\alpha\in I}\Delta_\alpha)$ we set $g = g_0$. Clearly g is a smooth metric on M, compatible with ω. We claim that the metric g will have arbitrarily large λ_1, provided that we took R to be small enough, and then picked ϵ, δ' to be sufficiently small, and $\hat{a}(\cdot)$ to be sufficiently large on $[R+\delta', 1-\delta']$. Let us show this.

Let $f : M \to \mathbb{R}$ be a smooth function with

$$\int_M f \, dg^{2n} = 0,$$

and

$$\int_M \|\nabla_g f\|_g^2 \, dg^{2n} \leqslant 1.$$

Then, for any $\alpha \in I$, define $f_\alpha : \overline{B}_1^{2n}(0) \to \mathbb{R}$ as $f_\alpha = (\Psi_\alpha)_* f$ - the pushforward of f by Ψ_α. Then keeping in mind that $\omega_\alpha = \omega_{std}$ on $B_R^{2n}(0)$, and that $g_\alpha = g_{std}$ on $B_r^{2n}(0)$, we get that

$$\begin{aligned} \int_{B_r^{2n}(0)} |\nabla f_\alpha|^2 \, dg_{std}^{2n} &= \int_{B_r^{2n}(0)} \|\nabla_{g_\alpha} f_\alpha\|_{g_\alpha}^2 \, dg_\alpha^{2n} \\ \leqslant \int_{B_1^{2n}(0)} \|\nabla_{g_\alpha} f_\alpha\|_{g_\alpha}^2 \, dg_\alpha^{2n} = \int_{\Delta_\alpha} \|\nabla_g f\|_g^2 \, dg^{2n} &\leqslant \int_M \|\nabla_g f\|_g^2 \, dg^{2n} \leqslant 1, \end{aligned} \tag{5.1}$$

that

$$\begin{aligned} \int_{D_{r,R}^{2n}} \|\nabla_{g_{loc}} f_\alpha\|_{g_{loc}}^2 \, dg_{std}^{2n} &= \int_{D_{r,R}^{2n}} \|\nabla_{g_\alpha} f_\alpha\|_{g_\alpha}^2 \, dg_\alpha^{2n} \\ \leqslant \int_{B_1^{2n}(0)} \|\nabla_{g_\alpha} f_\alpha\|_{g_\alpha}^2 \, dg_\alpha^{2n} = \int_{\Delta_\alpha} \|\nabla_g f\|_g^2 \, dg^{2n} &\leqslant \int_M \|\nabla_g f\|_g^2 \, dg^{2n} \leqslant 1, \end{aligned} \tag{5.2}$$

and that

$$\begin{aligned} \int_{D_{R,1}^{2n}} \|\nabla_{g_\alpha} f_\alpha\|_{g_\alpha}^2 \, dg_{std}^{2n} \leqslant C^{2n} \int_{D_{R,1}^{2n}} \|\nabla_{g_\alpha} f_\alpha\|_{g_\alpha}^2 \, dg_{0,\alpha}^{2n} &= C^{2n} \int_{D_{R,1}^{2n}} \|\nabla_{g_\alpha} f_\alpha\|_{g_\alpha}^2 \, dg_\alpha^{2n} \\ \leqslant C^{2n} \int_{B_1^{2n}(0)} \|\nabla_{g_\alpha} f_\alpha\|_{g_\alpha}^2 \, dg_\alpha^{2n} &= C^{2n} \int_{\Delta_\alpha} \|\nabla_g f\|_g^2 \, dg^{2n} \\ &\leqslant C^{2n} \int_M \|\nabla_g f\|_g^2 \, dg^{2n} \leqslant C^{2n}. \end{aligned} \tag{5.3}$$

Applying Proposition 1.11 to (5.2), we conclude that there exists some $E_\alpha \in \mathbb{R}$, such that for any $u \in (r, R)$ we have

$$\int_{S_u^{2n-1}} |f_\alpha - E_\alpha|^2 \, dg_{std}^{2n-1} \leqslant \epsilon, \tag{5.4}$$

which implies that

$$\begin{aligned} \int_{S^{2n-1}} |f_\alpha(u,\theta) - E_\alpha|^2 \, d\theta &= \frac{1}{u^{2n-1}} \int_{S_u^{2n-1}} |f_\alpha - E_\alpha|^2 \, dg_{std}^{2n-1} \\ &\leqslant \frac{\epsilon}{u^{2n-1}} \leqslant \frac{\epsilon}{r^{2n-1}} \leqslant \frac{2^{2n-1}\epsilon}{R^{2n-1}}, \end{aligned} \tag{5.5}$$

for any $u \in (r, R)$. Note that by a continuity reason, (5.4) and (5.5) hold also for $u = r, R$.

Applying our claim above to (5.3), we conclude that for any $R < u_1 < u_2 < 1$ we have

$$\int_{S^{2n-1}} |f_\alpha(u_2,\theta) - f_\alpha(u_1,\theta)|^2 \, d\theta \leqslant C^{2n} C' \delta'. \tag{5.6}$$

By a continuity reason, (5.6) holds for any $R \leqslant u_1 \leqslant u_2 \leqslant 1$.

We have that (5.5) is valid for $u = R$, and (5.6) holds when $u_1 = R$ and $R \leqslant u_2 \leqslant 1$. Hence we conclude that for any $u \in [R, 1]$ we have

$$\begin{aligned} &\int_{S^{2n-1}} |f_\alpha(u,\theta) - E_\alpha|^2 \, d\theta \\ \leqslant 2\Big(\int_{S^{2n-1}} |f_\alpha(u,\theta) - f_\alpha(R,\theta)|^2 \, d\theta &+ \int_{S^{2n-1}} |f_\alpha(R,\theta) - E_\alpha|^2 \, d\theta \Big) \\ &\leqslant 2C^{2n} C' \delta' + \frac{2^{2n}}{R^{2n-1}} \epsilon. \end{aligned} \tag{5.7}$$

Therefore from (5.5) and (5.7) we conclude that

$$\int_{S^{2n-1}} |f_\alpha(u,\theta) - E_\alpha|^2 \, d\theta \leqslant 2C^{2n} C' \delta' + \frac{2^{2n}}{R^{2n-1}} \epsilon, \tag{5.8}$$

for any $u \in [r, 1]$. This, in turn, implies that

$$\begin{aligned} \int_{S_u^{2n-1}} |f_\alpha - E_\alpha|^2 \, dg_{std}^{2n-1} &= u^{2n-1} \int_{S^{2n-1}} |f_\alpha(u,\theta) - E_\alpha|^2 \, d\theta \\ &\leqslant u^{2n-1} \left(2C^{2n} C' \delta' + \frac{2^{2n}}{R^{2n-1}} \epsilon \right) \leqslant 2C^{2n} C' \delta' + \frac{2^{2n}}{R^{2n-1}} \epsilon, \end{aligned} \tag{5.9}$$

for any $u \in [r, 1]$. Hence on one hand, from (5.9) we get

$$
\begin{aligned}
\int_{D^{2n}_{r,1}} |f_\alpha - E_\alpha|^2 \, dg^{2n}_{std} &= \int_r^1 \int_{S^{2n-1}_u} |f_\alpha - E_\alpha|^2 \, dg^{2n-1}_{std} \, du \\
\leqslant (1-r)\left(2C^{2n}C'\delta' + \frac{2^{2n}}{R^{2n-1}}\epsilon\right) &\leqslant 2C^{2n}C'\delta' + \frac{2^{2n}}{R^{2n-1}}\epsilon.
\end{aligned}
\tag{5.10}
$$

On the other hand, since (5.4) is true for $u = r$, and since we have (5.1), from Lemma 3.5 (section 3) we get

$$
\int_{B^{2n}_r(0)} |f_\alpha - E_\alpha|^2 dg^{2n}_{std} \leqslant C''(r^2 + \epsilon r) \leqslant C''(R^2 + \epsilon R),
\tag{5.11}
$$

where $C'' = C''(n)$.

Adding (5.10) and (5.11), we obtain

$$
\begin{aligned}
\int_{B^{2n}_1(0)} |f_\alpha - E_\alpha|^2 dg^{2n}_{std} = \int_{B^{2n}_r(0)} |f_\alpha - E_\alpha|^2 dg^{2n}_{std} &+ \int_{D^{2n}_{r,1}} |f_\alpha - E_\alpha|^2 \, dg^{2n}_{std} \\
\leqslant C''(R^2 + \epsilon R) + 2C^{2n}C'\delta' &+ \frac{2^{2n}}{R^{2n-1}}\epsilon \\
= C''R^2 + C''\epsilon R + 2C^{2n}C'\delta' &+ \frac{2^{2n}}{R^{2n-1}}\epsilon.
\end{aligned}
\tag{5.12}
$$

Look now at (5.7) and (5.12). We can choose ϵ and δ' to be small enough, so that we will have

$$
C''\epsilon R + 2C^{2n}C'\delta' + \frac{2^{2n}}{R^{2n-1}}\epsilon \leqslant C''R^2.
$$

Hence if we choose ϵ and δ' small, then (5.7) for the case of $u = 1$, and (5.12), will give us

$$
\int_{S^{2n-1}} |f_\alpha - E_\alpha|^2 \, dg^{2n-1}_{std} = \int_{S^{2n-1}} |f_\alpha(1,\theta) - E_\alpha|^2 \, d\theta \leqslant C''R^2,
\tag{5.13}
$$

and

$$
\int_{B^{2n}_1(0)} |f_\alpha - E_\alpha|^2 dg^{2n}_{std} \leqslant 2C''R^2.
\tag{5.14}
$$

Returning to the manifold M, from (5.13) and (5.14) we get

$$
\begin{aligned}
\int_{\partial\Delta_\alpha} |f - E_\alpha|^2 dg^{2n-1}_0 &= \int_{S^{2n-1}} |f_\alpha - E_\alpha|^2 dg^{2n-1}_{0,\alpha} \\
\leqslant C^{2n-1} \int_{S^{2n-1}} |f_\alpha - E_\alpha|^2 dg^{2n-1}_{std} &\leqslant C^{2n-1}C''R^2,
\end{aligned}
\tag{5.15}
$$

and

$$\begin{aligned}
&\int_{\Delta_\alpha} |f - E_\alpha|^2 dg_0^{2n} = \int_{B_1^{2n}(0)} |f_\alpha - E_\alpha|^2 dg_{0,\alpha}^{2n} \\
&\leqslant C^{2n} \int_{B_1^{2n}(0)} |f_\alpha - E_\alpha|^2 dg_{std}^{2n} \leqslant 2C^{2n}C'' R^2.
\end{aligned} \tag{5.16}$$

Now consider two adjacent simplices Δ_α and Δ_β, having a common face which we denote by $\Sigma \subset M$. Then (5.15) implies

$$\int_\Sigma |f - E_\alpha|^2 dg_0^{2n-1} \leqslant \int_{\partial \Delta_\alpha} |f - E_\alpha|^2 dg_0^{2n-1} \leqslant C^{2n-1} C'' R^2,$$

and

$$\int_\Sigma |f - E_\beta|^2 dg_0^{2n-1} \leqslant \int_{\partial \Delta_\beta} |f - E_\beta|^2 dg_0^{2n-1} \leqslant C^{2n-1} C'' R^2.$$

Therefore,

$$\begin{aligned}
Vol_{g_0}(\Sigma) |E_\alpha - E_\beta|^2 &= \int_\Sigma |E_\alpha - E_\beta|^2 dg_0^{2n-1} \\
\leqslant 2\Big(\int_\Sigma |f - E_\alpha|^2 dg_0^{2n-1} &+ \int_\Sigma |f - E_\beta|^2 dg_0^{2n-1} \Big) \leqslant 4C^{2n-1} C'' R^2,
\end{aligned}$$

Since we have only finite number of faces of simplices of our triangulation, it follows that the minimum of a g_0-volume of such a face, is a positive real number. Denote it by $c > 0$. Hence we get the following: if Δ_α and Δ_β, where $\alpha, \beta \in I$, are adjacent simplices from our triangulation, then

$$|E_\alpha - E_\beta|^2 \leqslant \frac{4C^{2n-1} C''}{c} R^2.$$

Now, if we consider any two simplices Δ_α and Δ_β (not necessarily adjacent), then we can connect Δ_α with Δ_β via a sequence of distinct simplices from our triangulation, where any two consequent simplices in this sequence are adjacent, and hence by the triangle inequality we get

$$|E_\alpha - E_\beta|^2 \leqslant \frac{4|I|^2 C^{2n-1} C''}{c} R^2,$$

for any $\alpha, \beta \in I$. Therefore there exists some $E \in \mathbb{R}$ such that

$$|E_\alpha - E|^2 \leqslant \frac{4|I|^2 C^{2n-1} C''}{c} R^2, \tag{5.17}$$

for any $\alpha \in I$ (we can just take $E = E_\gamma$ for any $\gamma \in I$).

Therefore, from (5.16) and (5.17) we get

$$\begin{aligned}\int_{\Delta_\alpha} |f - E|^2 dg_0^{2n} &\leqslant 2\bigg(\int_{\Delta_\alpha} |f - E_\alpha|^2 dg_0^{2n} + |E_\alpha - E|^2 Vol_{g_0}(\Delta_\alpha)\bigg) \\ &\leqslant 4C^{2n}C''R^2 + \frac{8|I|^2 C^{2n-1} C'' Vol_{g_0}(\Delta_\alpha)}{c} R^2\end{aligned} \tag{5.18}$$

Summing (5.18) over all $\alpha \in I$, we get

$$\begin{aligned}\int_M |f - E|^2 dg_0^{2n} &= \sum_{\alpha \in I} \int_{\Delta_\alpha} |f - E|^2 dg_0^{2n} \\ &\leqslant 4|I|C^{2n}C''R^2 + \frac{8|I|^2 C^{2n-1} C'' Vol_{g_0}(M)}{c} R^2 \\ &= \left(4|I|C^{2n}C'' + \frac{8|I|^2 C^{2n-1} C'' Vol_{g_0}(M)}{c}\right) R^2\end{aligned}$$

Note that

$$\int_M |f|^2 dg^{2n} \leqslant \int_M |f|^2 dg^{2n} + E^2 = \int_M |f - E|^2 dg^{2n} = \int_M |f - E|^2 dg_0^{2n}.$$

Therefore

$$\int_M |f|^2 dg^{2n} \leqslant C''' R^2,$$

where

$$C''' = 4|I|C^{2n}C'' + \frac{8|I|^2 C^{2n-1} C'' Vol_{g_0}(M)}{c}.$$

Hence we have finally proved the following:

If $f : M \to \mathbb{R}$ is a smooth function with

$$\int_M f \, dg^{2n} = 0,$$

and

$$\int_M \|\nabla_g f\|_g^2 \, dg^{2n} \leqslant 1,$$

then

$$\int_M |f|^2 dg^{2n} \leqslant C''' R^2.$$

Therefore we immediately get a lower bound for the first eigenvalue:

$$\lambda_1(g) \geqslant \frac{1}{C''' R^2}.$$

Note that the constant C''' depends only on M, on the metric g_0 on M, on our triangulation of M to simplices Δ_α, and on the collection of maps $\Psi_\alpha : \overline{\Delta_\alpha} \to \overline{B}_1^{2n}(0)$. Therefore, since we have freedom to choose $R > 0$ to be arbitrarily small, this means that λ_1 associated with the metric g, can be arbitrarily large.

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